# Single-Particle Motion in Given Electric and Magnetic Fields

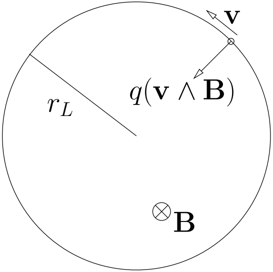
I H Hutchinson

Plasma Science and Fusion Center,
and Department of Nuclear Science and Engineering
Massachusetts Institute of Technology, Cambridge, MA, USA

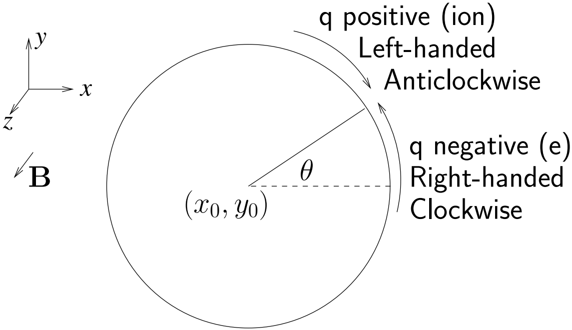
Introduction to Plasma Physics, 2017

# Basic circular orbit

First the simple circular orbit in uniform $B$.




$B$ into page.
Centripetal force.
$qvB = mv^2/r_L \Rightarrow$
$r_L = \frac{mv}{qB}$   $\Omega = \frac{qB}{m}$




$B$ out of page. Direction of rotation depends on sign of charge.

Plus constant uniform motion along the field → helical orbit.

# Drifts arising from $E$ and non-uniform $B$

Additional movement arises when either

there is an electric field $\boldsymbol{E}$

or else $\boldsymbol{B}$ is not simply uniform.

These are treated by considering the motion to consist of

- Gyration in a circular orbit, plus...
- Drift of the center of the circular orbit.

The drifts can be calculated one at a time and then added together.

Alternatively, they can be derived all together and show that the particle behaves like a composite entity that has both:

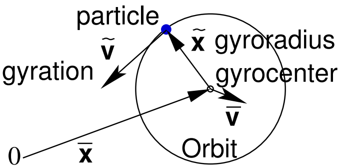
- Charge $q$, and
- Magnetic moment $\boldsymbol{\mu}$.

# Express the particle motion as sum of two parts

## Average over one gyro-orbit is the gyrocenter. Gyrating part is the gyroradius.

Divide position and velocity into average and gyrating parts: $\bar{\boldsymbol{x}}$ and $\tilde{\boldsymbol{x}}$, $\bar{\boldsymbol{v}}$ and $\tilde{\boldsymbol{v}}$. ($\bar{\boldsymbol{v}} = \langle \boldsymbol{v} \rangle$, and $\langle \tilde{\boldsymbol{v}} \rangle = 0$ etc.) then:

$$\boldsymbol{x} = \bar{\boldsymbol{x}} + \tilde{\boldsymbol{x}} \quad \text{and} \quad \boldsymbol{v} = \tilde{\boldsymbol{v}} + \bar{\boldsymbol{v}}$$




The gyrocenter is $\bar{\boldsymbol{x}}$, and moves with velocity $\bar{\boldsymbol{v}}$. But typical $\tilde{\boldsymbol{v}} \gg \bar{\boldsymbol{v}}$.

If $\boldsymbol{B}$ is nonuniform then its value at the particle also has average and oscillating parts:

$$\boldsymbol{B}(x) = \boldsymbol{B}(\bar{x} + \tilde{x}) = \bar{\boldsymbol{B}} + \tilde{\boldsymbol{B}} = \boldsymbol{B}(\bar{x}) + (\tilde{x}.\nabla)\boldsymbol{B} + O(\tilde{x}^2)$$

And similarly for $\boldsymbol{E} = \bar{\boldsymbol{E}} + \tilde{\boldsymbol{E}}$. Note: $\bar{\boldsymbol{v}}/\tilde{\boldsymbol{v}} \sim \tilde{\boldsymbol{B}}/\bar{\boldsymbol{B}} \sim \tilde{\boldsymbol{E}}/\bar{\boldsymbol{E}} \sim \epsilon$.

Equation of motion of particle charge $q$, mass $m$ is:

$$\frac{d}{dt} m\boldsymbol{v} = q(\boldsymbol{E} + \boldsymbol{v} \wedge \boldsymbol{B})$$

# Substitute parts into the equation of motion

$$\frac{d}{dt}m(\bar{\boldsymbol{v}}+\tilde{\boldsymbol{v}}) = q[\boldsymbol{E}+(\bar{\boldsymbol{v}}+\tilde{\boldsymbol{v}})\wedge(\bar{\boldsymbol{B}}+\tilde{\boldsymbol{B}})] \qquad \boxed{\text{Total}}$$

$$= q[\boldsymbol{E}+\tilde{\boldsymbol{v}}\wedge\bar{\boldsymbol{B}}+\tilde{\boldsymbol{v}}\wedge\tilde{\boldsymbol{B}}+\bar{\boldsymbol{v}}\wedge\bar{\boldsymbol{B}}+\bar{\boldsymbol{v}}\wedge\tilde{\boldsymbol{B}}] \qquad (1)$$

Average equation over gyroperiod (remember $\langle\ \tilde{}\ \rangle = 0$):

$$\frac{d}{dt}m\bar{\boldsymbol{v}} = q[\bar{\boldsymbol{E}}+\bar{\boldsymbol{v}}\wedge\bar{\boldsymbol{B}}+\langle\tilde{\boldsymbol{v}}\wedge\tilde{\boldsymbol{B}}\rangle] \qquad \boxed{\text{Averaged}} \qquad (2)$$

Subtract from $\boxed{\text{Total}}$ equation (1) to get $\boxed{\text{Fluctuating}}$

$$\underbrace{\frac{d}{dt}m\tilde{\boldsymbol{v}}}_{\epsilon^0} = q[\underbrace{\tilde{\boldsymbol{E}}}_{\epsilon}+\underbrace{\tilde{\boldsymbol{v}}\wedge\bar{\boldsymbol{B}}}_{\epsilon^0}+\underbrace{\tilde{\boldsymbol{v}}\wedge\tilde{\boldsymbol{B}}}_{\epsilon}-\underbrace{\langle\tilde{\boldsymbol{v}}\wedge\tilde{\boldsymbol{B}}\rangle}_{\epsilon}+\underbrace{\bar{\boldsymbol{v}}\wedge\tilde{\boldsymbol{B}}}_{\epsilon^2}] \qquad (3)$$

Solve the zeroth order terms of Fluctuating equation
$\frac{d}{dt}m\tilde{\boldsymbol{v}} = q[\tilde{\boldsymbol{v}}\wedge\bar{\boldsymbol{B}}]$
To get circular gyro-orbit: $\tilde{\boldsymbol{v}} = \frac{d}{dt}\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{x}}\wedge q\bar{\boldsymbol{B}}/m$ (obviously).

# Evaluate the doubly-fluctuating term $\langle \tilde{\mathbf{v}} \wedge \tilde{\mathbf{B}} \rangle$

The term we want is $\tilde{\mathbf{v}} \wedge \tilde{\mathbf{B}} = \tilde{\mathbf{v}} \wedge (\tilde{\mathbf{x}}.\nabla)\mathbf{B}$.

But (for $\nabla.\mathbf{B} = 0$) $\nabla[(\tilde{\mathbf{v}} \wedge \tilde{\mathbf{x}}).\mathbf{B}] = \tilde{\mathbf{v}} \wedge (\tilde{\mathbf{x}}.\nabla)\mathbf{B} - \tilde{\mathbf{x}} \wedge (\tilde{\mathbf{v}}.\nabla)\mathbf{B}$.

Also, since $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{x}}$ rotate perpendicular, in circles, at frequency $\Omega$, $\Omega\tilde{\mathbf{x}}(t + \frac{\pi}{2|\Omega|}) = \tilde{\mathbf{v}}(t)$ and $\tilde{\mathbf{v}}(t + \frac{\pi}{2|\Omega|}) = -\Omega\tilde{\mathbf{x}}(t)$. Therefore

$$\langle \tilde{\mathbf{v}} \wedge (\tilde{\mathbf{x}}.\nabla)\mathbf{B} \rangle = -\langle \tilde{\mathbf{x}} \wedge (\tilde{\mathbf{v}}.\nabla)\mathbf{B} \rangle.$$

Hence

$$\langle \tilde{\mathbf{v}} \wedge \tilde{\mathbf{B}} \rangle = \langle \tilde{\mathbf{v}} \wedge (\tilde{\mathbf{x}}.\nabla)\mathbf{B} \rangle = \nabla[\langle \tfrac{1}{2}(\tilde{\mathbf{v}} \wedge \tilde{\mathbf{x}}) \rangle.\mathbf{B}].$$

Now the quantity $\frac{1}{2}\langle \tilde{\mathbf{v}} \wedge \tilde{\mathbf{x}} \rangle$ is geometrically the rate of sweeping out area by the gyro-radius $\tilde{\mathbf{x}}$.

Thus, the magnetic moment of the gyro-orbit is $\boldsymbol{\mu} \equiv \frac{1}{2}q\langle \tilde{\mathbf{v}} \wedge \tilde{\mathbf{x}} \rangle$.

So $q\langle \tilde{\mathbf{v}} \wedge \tilde{\mathbf{B}} \rangle = \nabla(\boldsymbol{\mu}.\mathbf{B})$, and the Averaged equation becomes

$$\boxed{\frac{d}{dt}m\bar{\mathbf{v}} = q[\bar{\mathbf{E}} + \bar{\mathbf{v}} \wedge \bar{\mathbf{B}}] + \nabla(\boldsymbol{\mu}.\mathbf{B})}$$

This is the equation of motion of the gyrocenter: the "drift orbit".

# Additional details of $\langle \tilde{\boldsymbol{v}} \wedge \tilde{B} \rangle$ evaluation

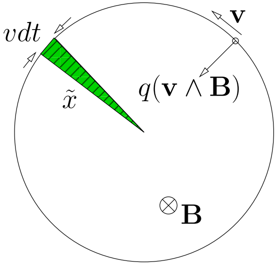
Vector identity

$$\boldsymbol{w}[\boldsymbol{x}.(\boldsymbol{y} \wedge \boldsymbol{z})] = (\boldsymbol{w}.\boldsymbol{x})(\boldsymbol{y} \wedge \boldsymbol{z}) + (\boldsymbol{w}.\boldsymbol{y})(\boldsymbol{z} \wedge \boldsymbol{x}) + (\boldsymbol{w}.\boldsymbol{z})(\boldsymbol{x} \wedge \boldsymbol{y})$$

Applied using $\nabla.\boldsymbol{B} = 0$: $\nabla[\boldsymbol{B}.(\tilde{\boldsymbol{v}} \wedge \tilde{\boldsymbol{x}})] = \tilde{\boldsymbol{v}} \wedge (\tilde{\boldsymbol{x}}.\boldsymbol{\nabla})\boldsymbol{B} - \tilde{\boldsymbol{x}} \wedge (\tilde{\boldsymbol{v}}.\boldsymbol{\nabla})\boldsymbol{B}$
(Gradient applies only to $\boldsymbol{B}$).
The two terms on RHS are *equal* when averaged.
So each is equal to half LHS average.

$$\langle \tilde{\boldsymbol{v}} \wedge \tilde{\boldsymbol{B}} \rangle = \langle \tilde{\boldsymbol{v}} \wedge (\tilde{\boldsymbol{x}}.\boldsymbol{\nabla})\boldsymbol{B} \rangle = \boldsymbol{\nabla}[\langle \tfrac{1}{2}(\tilde{\boldsymbol{v}} \wedge \tilde{\boldsymbol{x}}) \rangle.\boldsymbol{B}].$$

Area $\pi\tilde{x}^2$ times average current round orbit
is magnetic moment, $\mu = \frac{1}{2}mv_\perp^2/B$.
That is Area$\times$Charge/time = Area/time$\times q$
Area/time is $\frac{1}{2}v\tilde{x} = \frac{1}{2}\langle \tilde{\boldsymbol{v}} \wedge \tilde{\boldsymbol{x}} \rangle$
Thus $\boldsymbol{\mu} = \frac{1}{2}\langle \tilde{\boldsymbol{v}} \wedge \tilde{\boldsymbol{x}} \rangle q$.



# Energy and magnetic moment are conserved

naturally by the equations. They are constants of the motion.

Take $\bar{\boldsymbol{v}}.$ Averaged to get the Center Energy equation:

$$\bar{\boldsymbol{v}}.\frac{d}{dt}m\bar{\boldsymbol{v}} = \frac{d}{dt}(\frac{1}{2}m\bar{\boldsymbol{v}}^2) = q\bar{\boldsymbol{v}}.\bar{\boldsymbol{E}} + \bar{\boldsymbol{v}}.\nabla(\boldsymbol{\mu}.\boldsymbol{B})$$

and $\langle\tilde{\boldsymbol{v}}.$ Fluctuating$\rangle$ equations to get the: Gyration Energy:

$$\langle\frac{d}{dt}(\frac{1}{2}m\tilde{\boldsymbol{v}}^2)\rangle = q[\langle\tilde{\boldsymbol{v}}.\tilde{\boldsymbol{E}}\rangle + \langle\tilde{\boldsymbol{v}}.(\bar{\boldsymbol{v}}\wedge\tilde{\boldsymbol{B}})\rangle] = q\langle\tilde{\boldsymbol{v}}.\tilde{\boldsymbol{E}}\rangle - \bar{\boldsymbol{v}}.\nabla(\boldsymbol{\mu}.\boldsymbol{B})$$

The sum of these equations is total Conservation of Energy: $\frac{d}{dt}[\frac{1}{2}m(\bar{\boldsymbol{v}}^2 + \langle\tilde{\boldsymbol{v}}^2\rangle)] = q[\bar{\boldsymbol{v}}.\bar{\boldsymbol{E}} + \langle\tilde{\boldsymbol{v}}.\tilde{\boldsymbol{E}}\rangle]$

Only $\boldsymbol{E}$ does work on the particle. However one can show from Faraday's law of induction that $q\langle\tilde{\boldsymbol{v}}.\tilde{\boldsymbol{E}}\rangle = -\boldsymbol{\mu}.\partial\bar{\boldsymbol{B}}/\partial t = \mu\partial B/\partial t$.

Then using $\mu = \frac{1}{2}m\tilde{v}^2/B$ the Gyration Energy equation can be written $\frac{d}{dt}(\mu B) = \mu[\frac{\partial}{\partial t} + \bar{\boldsymbol{v}}.\nabla]B = \mu\frac{d}{dt}B \quad \Rightarrow \quad \mu = \frac{mv_\perp^2}{2B} = const.$

Magnetic moment $\mu$ is a constant of the motion.

# Various drifts arise from Averaged GC Equation

Use $\frac{d}{dt}m\bar{\boldsymbol{v}} = q[\bar{\boldsymbol{E}} + \bar{\boldsymbol{v}} \wedge \bar{\boldsymbol{B}}] + \nabla(\boldsymbol{\mu}.\boldsymbol{B})$. Write $\bar{\boldsymbol{v}} = \boldsymbol{v}_\parallel + \boldsymbol{v}_d$.

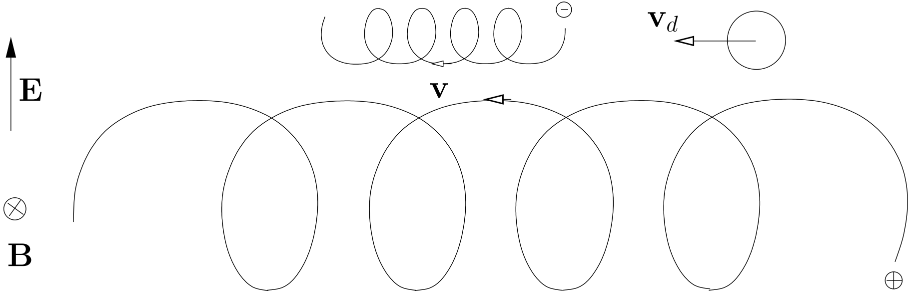
- Uniform $\boldsymbol{E}$ & $\boldsymbol{B}$. $0 = q[\boldsymbol{E} + \boldsymbol{v}_d \wedge \bar{\boldsymbol{B}}]$
  $\boldsymbol{E} \wedge \boldsymbol{B}$ drift: ......................... $\boldsymbol{v}_d = \boldsymbol{v}_E = (\boldsymbol{E} \wedge \bar{\boldsymbol{B}})/B^2$.
- Non-constant $\boldsymbol{E}$, uniform $B$. $\frac{d}{dt}m\boldsymbol{v}_E = q(\boldsymbol{v}_d \wedge \bar{\boldsymbol{B}})$
  Polarization drift: ................ $\boldsymbol{v}_d = \boldsymbol{v}_p = -\frac{d}{dt}m\boldsymbol{v}_E \wedge \bar{\boldsymbol{B}}/qB^2$
- Non-uniform $B$, $\boldsymbol{E} = 0$. $0 = q\boldsymbol{v}_d \wedge \boldsymbol{B} - \nabla_\perp(\mu B)$
  Grad-B drift: .................. $\boldsymbol{v}_d = \boldsymbol{v}_{\nabla B} = -\mu(\nabla B \wedge \bar{\boldsymbol{B}})/qB^2$
- Curved $\boldsymbol{B}$, $\boldsymbol{E} = 0$. $\frac{d}{dt}m\boldsymbol{v}_\parallel = v_\parallel(\frac{\boldsymbol{B}}{B}.\nabla)\frac{\boldsymbol{B}}{B}mv_\parallel = q(\boldsymbol{v}_d \wedge \bar{\boldsymbol{B}})$
  Curvature drift: ................. $\boldsymbol{v}_d = \boldsymbol{v}_\kappa = -mv_\parallel^2(\boldsymbol{\kappa} \wedge \bar{\boldsymbol{B}})/qB^2$
- Parallel Mirror Force: ...................... $\frac{d}{dt}mv_\parallel = qE_\parallel - \mu\nabla_\parallel B$

Total gyrocenter motion is the sum of these drifts.

# Intuitive understanding of drifts.

# $E \wedge B$ drift in crossed electric/magnetic fields

Orbit is wider where particle kinetic energy is greater.




Drift velocity $\boldsymbol{v}_E = \boldsymbol{E} \wedge \boldsymbol{B}/B^2$ same for all particles.

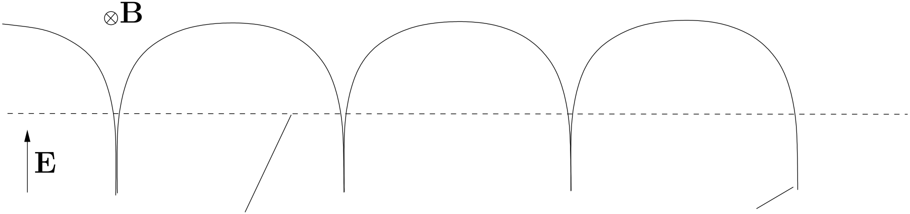
Can also be thought of as a shift to the frame of reference moving with velocity $\boldsymbol{v}_E$ in which $\boldsymbol{E} = 0$.

Any other force $\boldsymbol{F}$ (e.g. gravity) gives similar drift $\boldsymbol{F} \wedge \boldsymbol{B}/qB^2$.

# Polarization drift from non-constant $E$-field

Occurs only when $d\boldsymbol{E}/dt \neq 0$. A "displacement" more than drift.

Example. A finite $\boldsymbol{E}$ suddenly turned on, initially stationary particle.



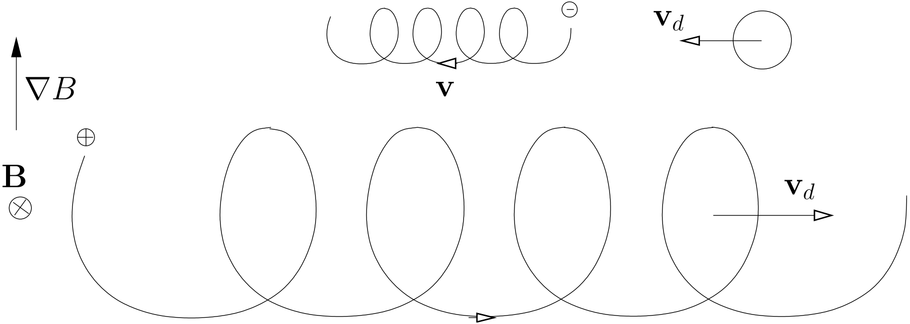

Final average (gyro) locus. Initial Position at rest (when $E = 0$).

$$\boldsymbol{v}_p = -\frac{d}{dt} m\boldsymbol{v}_E \wedge \boldsymbol{B}/qB^2 = -m(\frac{d}{dt}\boldsymbol{E} \wedge \boldsymbol{B}) \wedge \boldsymbol{B}/qB^4 = m\frac{d}{dt}\boldsymbol{E}_\perp/qB^2$$

Displacement $\Delta \boldsymbol{x} = \int \boldsymbol{v}_p dt = m\Delta\boldsymbol{E}_\perp/qB^2$

# $\nabla B$ drift from field-strength gradient

Orbit is tighter where $B$ is bigger. Leads to sideways drift.

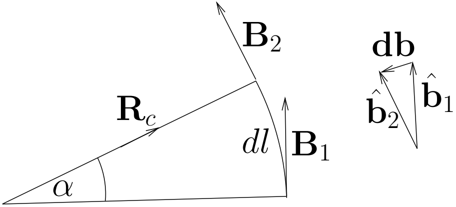



Opposite charges gyrate opposite directions; hence opposite drifts.
Speed depends on $v_\perp^2$ and hence $\mu$.

Grad-B drift $\quad \boldsymbol{v}_{\nabla B} = -\mu(\boldsymbol{\nabla} B \wedge \bar{\boldsymbol{B}})/qB^2$

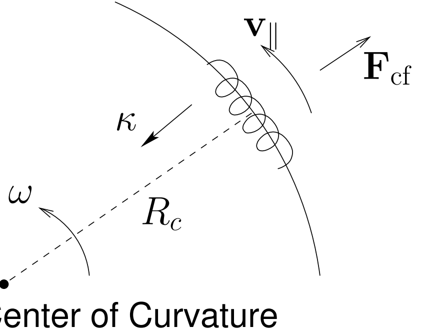
# Curvature drift from centrifugal force

Definition of curvature
$\boldsymbol{\kappa} \equiv \frac{d\boldsymbol{b}}{dl} = \boldsymbol{b}.\nabla\boldsymbol{b}$ where $\boldsymbol{b} = \boldsymbol{B}/B$.




Radius of curvature $\boldsymbol{R}_c = -\boldsymbol{\kappa}/\kappa^2$

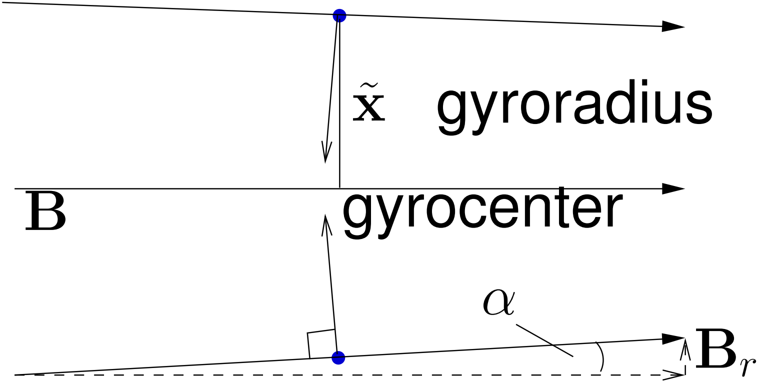
Centrifugal force $\boldsymbol{F}_{cf} = mv_\parallel^2 \frac{\boldsymbol{R}_c}{R_c^2}$.




Resulting curvature drift $\boldsymbol{v}_\kappa = \boldsymbol{F}_{cf} \wedge \boldsymbol{B}/qB^2 = -mv_\parallel^2(\boldsymbol{\kappa} \wedge \bar{\boldsymbol{B}})/qB^2$

# Parallel $\nabla(\mu B)$ mirror force from converging field.

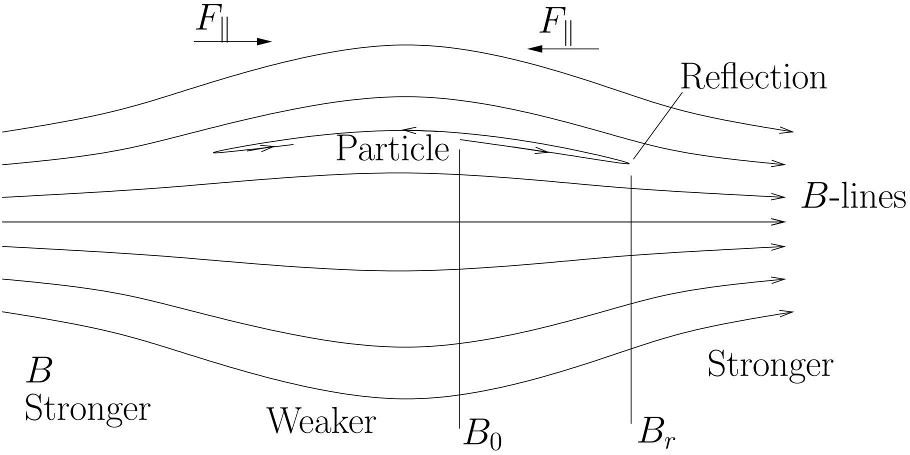
Mirror force arises from the component of the magnetic force in the direction $\boldsymbol{B}(\bar{\boldsymbol{x}})$ (horizontal here) averaged over a gyro-orbit.



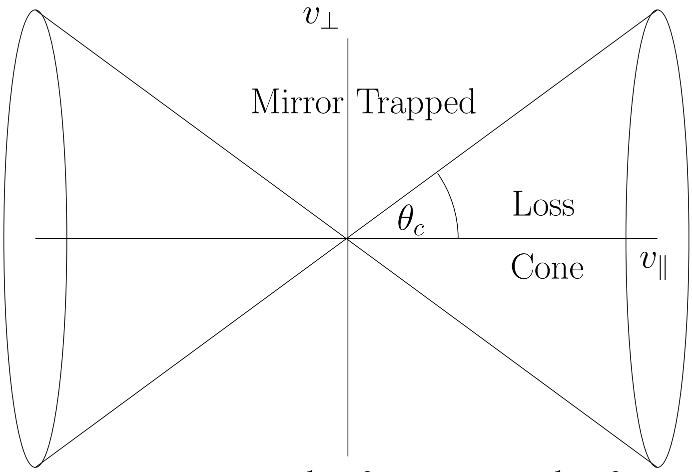

Directed away from stronger-B regions. ($\mu$ is antiparallel to $\boldsymbol{B}$.)

# Magnetic Mirror traps some particles

Particles with sufficient $v_\perp$ are reflected from high-B and bounce.



# Trapping is determined by the velocity pitch angle




Velocity space at the place where the field is $B$.

Constancy of $\mu$ means $\frac{1}{2}mv_\perp^2 = \mu B$. Also $\frac{1}{2}mv^2 = const$.
If max field is $B_m$, particle is trapped only if
$\mu B_m = \frac{1}{2}mv_\perp^2 B_m/B > \frac{1}{2}m(v_\parallel^2 + v_\perp^2)$. i.e. $\sin^2\theta > \sin^2\theta_c = B/B_m$.