Single-Particle Motion in Given Electric and Magnetic Fields

I H Hutchinson

Plasma Science and Fusion Center,
and Department of Nuclear Science and Engineering
Massachusetts Institute of Technology, Cambridge, MA, USA

Introduction to Plasma Physics, 2017
Basic circular orbit

First the simple circular orbit in uniform $B$.

$$qvB = \frac{mv^2}{r_L} \Rightarrow r_L = \frac{mv}{qB} \quad \Omega = \frac{qB}{m}$$

$B$ into page. Centripetal force.

$B$ out of page. Direction of rotation depends on sign of charge.

Plus constant uniform motion along the field $\rightarrow$ helical orbit.
Drifts arising from \( E \) and non-uniform \( B \)

Additional movement arises when either
there is an electric field \( E \)
or else \( B \) is not simply uniform.

These are treated by considering the motion to consist of
- Gyration in a circular orbit, plus...
- Drift of the center of the circular orbit.

The drifts can be calculated one at a time and then added together.

Alternatively, they can be derived all together and show that the particle behaves like a composite entity that has both:
- Charge \( q \), and
- Magnetic moment \( \mu \).
Express the particle motion as sum of two parts

Average over one gyro-orbit is the gyrocenter. Gyrating part is the gyroradius.

Divide position and velocity into average and gyrating parts: \( \bar{x} \) and \( \tilde{x} \), \( \bar{v} \) and \( \tilde{v} \).

\( \bar{v} = \langle v \rangle \), and \( \langle \tilde{v} \rangle = 0 \) etc.

then:

\[
\begin{align*}
\mathbf{x} &= \bar{x} + \tilde{x} \quad \text{and} \quad \mathbf{v} = \tilde{v} + \bar{v} \\
\end{align*}
\]

The gyrocenter is \( \bar{x} \), and moves with velocity \( \bar{v} \). But typical \( \tilde{v} \gg \bar{v} \).

If \( B \) is nonuniform then its value at the particle also has average and oscillating parts:

\[
B(\mathbf{x}) = B(\bar{x} + \tilde{x}) = \bar{B} + \tilde{B} = B(\bar{x}) + (\tilde{x} \cdot \nabla)B + O(\tilde{x}^2)
\]

And similarly for \( E = \bar{E} + \tilde{E} \). Note: \( \tilde{v}/\bar{v} \sim \tilde{B}/\bar{B} \sim \tilde{E}/\bar{E} \sim \epsilon \).

Equation of motion of particle charge \( q \), mass \( m \) is:

\[
\frac{d}{dt} m\mathbf{v} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})
\]
Substitute parts into the equation of motion

\[
\frac{d}{dt} m(\vec{v} + \tilde{v}) = q[E + (\vec{v} + \tilde{v}) \wedge (\vec{B} + \tilde{B})]
\]

Total

\[
= q[E + \tilde{v} \wedge \vec{B} + \tilde{v} \wedge \vec{B} + \vec{v} \wedge \vec{B} + \vec{v} \wedge \vec{B}]
\]

(1)

Average equation over gyroperiod (remember \(\langle \tilde{v} \rangle = 0\)):

\[
\frac{d}{dt} m\tilde{v} = q[\bar{E} + \tilde{v} \wedge \vec{B} + \langle \tilde{v} \wedge \vec{B} \rangle]
\]

Averaged

(2)

Subtract from Total equation (1) to get Fluctuating

\[
\frac{d}{dt} m\vec{v} = q[\bar{E} + \tilde{v} \wedge \vec{B} + \langle \tilde{v} \wedge \vec{B} \rangle - \langle \tilde{v} \wedge \vec{B} \rangle + \vec{v} \wedge \vec{B}]
\]

(3)

Solve the zeroth order terms of Fluctuating equation

\[
\frac{d}{dt} m\vec{v} = q[\tilde{v} \wedge \vec{B}]
\]

To get circular gyro-orbit: \(\tilde{v} = \frac{d}{dt}\tilde{x} = \tilde{x} \wedge q\vec{B}/m\) (obviously).
Evaluate the doubly-fluctuating term $\langle \tilde{v} \wedge \tilde{B} \rangle$

The term we want is $\tilde{v} \wedge \tilde{B} = \tilde{v} \wedge (\tilde{x} \cdot \nabla)B$.

But (for $\nabla \cdot B = 0$) $\nabla[(\tilde{v} \wedge \tilde{x}) \cdot B] = \tilde{v} \wedge (\tilde{x} \cdot \nabla)B - \tilde{x} \wedge (\tilde{v} \cdot \nabla)B$.

Also, since $\tilde{v}$ and $\tilde{x}$ rotate perpendicular, in circles, at frequency $\Omega$, $\Omega \tilde{x}(t + \frac{\pi}{2|\Omega|}) = \tilde{v}(t)$ and $\tilde{v}(t + \frac{\pi}{2|\Omega|}) = -\Omega \tilde{x}(t)$. Therefore

$$\langle 	ilde{v} \wedge (\tilde{x} \cdot \nabla)B \rangle = -\langle \tilde{x} \wedge (\tilde{v} \cdot \nabla)B \rangle.$$

Hence

$$\langle \tilde{v} \wedge \tilde{B} \rangle = \langle \tilde{v} \wedge (\tilde{x} \cdot \nabla)B \rangle = \nabla[\langle \frac{1}{2}(\tilde{v} \wedge \tilde{x}) \rangle \cdot B].$$

Now the quantity $\frac{1}{2} \langle \tilde{v} \wedge \tilde{x} \rangle$ is geometrically the rate of sweeping out area by the gyro-radius $\tilde{x}$.

Thus, the magnetic moment of the gyro-orbit is $\mu \equiv \frac{1}{2} q \langle \tilde{v} \wedge \tilde{x} \rangle$.

So $q \langle \tilde{v} \wedge \tilde{B} \rangle = \nabla(\mu \cdot B)$, and the Averaged equation becomes

$$\frac{d}{dt} m\tilde{v} = q[\tilde{E} + \tilde{v} \wedge \tilde{B}] + \nabla(\mu \cdot B).$$

This is the equation of motion of the gyrocenter: the “drift orbit”.
Additional details of $\langle \tilde{v} \wedge \tilde{B} \rangle$ evaluation

Vector identity

$$w[x \cdot (y \wedge z)] = (w \cdot x)(y \wedge z) + (w \cdot y)(z \wedge x) + (w \cdot z)(x \wedge y)$$

Applied using $\nabla \cdot B = 0$: $\nabla[B \cdot (\tilde{v} \wedge \tilde{x})] = \tilde{v} \wedge (\tilde{x} \cdot \nabla)B - \tilde{x} \wedge (\tilde{v} \cdot \nabla)B$

(Gradient applies only to $B$).

The two terms on RHS are equal when averaged.

So each is equal to half LHS average.

$$\langle \tilde{v} \wedge \tilde{B} \rangle = \langle \tilde{v} \wedge (\tilde{x} \cdot \nabla)B \rangle = \nabla[\langle \frac{1}{2}(\tilde{v} \wedge \tilde{x}) \rangle].B].$$

Area $\pi \tilde{x}^2$ times average current round orbit is magnetic moment, $\mu = \frac{1}{2}mv_{\perp}^2/B$.

That is $\text{Area} \times \text{Charge/time} = \text{Area/time} \times q$

Area/time is $\frac{1}{2}v \tilde{x} = \frac{1}{2}\langle \tilde{v} \wedge \tilde{x} \rangle$

Thus $\mu = \frac{1}{2}\langle \tilde{v} \wedge \tilde{x} \rangle q$. 
Energy and magnetic moment are conserved naturally by the equations. They are constants of the motion.

Take $\bar{v}$ Averaged to get the Center Energy equation:

$$\bar{v} \cdot \frac{d}{dt} m\bar{v} = \frac{d}{dt} \left( \frac{1}{2} m\bar{v}^2 \right) = q\bar{v} \cdot \bar{E} + \bar{v} \cdot \nabla (\mu \cdot B)$$

and $\langle \tilde{v} \cdot \text{Fluctuating} \rangle$ equations to get the: Gyration Energy:

$$\langle \frac{d}{dt} \left( \frac{1}{2} m\tilde{v}^2 \right) \rangle = q[\langle \tilde{v} \cdot \bar{E} \rangle + \langle \tilde{v} \cdot (\bar{v} \wedge \tilde{B}) \rangle] = q\langle \tilde{v} \cdot \bar{E} \rangle - \bar{v} \cdot \nabla (\mu \cdot B)$$

The sum of these equations is total Conservation of Energy:

$$\frac{d}{dt} \left[ \frac{1}{2} m (\bar{v}^2 + \langle \tilde{v}^2 \rangle) \right] = q[\bar{v} \cdot \bar{E} + \langle \tilde{v} \cdot \bar{E} \rangle]$$

Only $E$ does work on the particle. However one can show from Faraday’s law of induction that $q\langle \tilde{v} \cdot \bar{E} \rangle = -\mu \partial \bar{B} / \partial t = \mu \partial B / \partial t$.

Then using $\mu = \frac{1}{2} m\tilde{v}^2 / B$ the Gyration Energy equation can be written

$$\frac{d}{dt} (\mu B) = \mu \left[ \frac{\partial}{\partial t} + \bar{v} \cdot \nabla \right] B = \mu \frac{d}{dt} B \quad \Rightarrow \quad \mu = \frac{mv^2}{2B} = \text{const.}$$

Magnetic moment $\mu$ is a constant of the motion.
Various drifts arise from Averaged GC Equation

Use \( \frac{d}{dt} m\vec{v} = q[\vec{E} + \vec{v} \wedge \vec{B}] + \nabla (\mu \cdot B) \). Write \( \vec{v} = \vec{v}_\parallel + \vec{v}_d \).

- **Uniform \( E \) & \( B \).** \( 0 = q[\vec{E} + \vec{v}_d \wedge \vec{B}] \)
  
  **\( E \wedge B \) drift:** \( \vec{v}_d = \vec{v}_E = (\vec{E} \wedge \vec{B})/B^2 \).

- **Non-constant \( E \), uniform \( B \).** \( \frac{d}{dt} mv_E = q(\vec{v}_d \wedge \vec{B}) \)
  
  **Polarization drift:** \( \vec{v}_d = \vec{v}_p = -\frac{d}{dt} mv_E \wedge \vec{B}/qB^2 \).

- **Non-uniform \( B \), \( E = 0 \).** \( 0 = q\vec{v}_d \wedge \vec{B} - \nabla (\mu B) \)
  
  **Grad-B drift:** \( \vec{v}_d = \vec{v}_{\nabla B} = -\mu(\nabla B \wedge \vec{B})/qB^2 \).

- **Curved \( B \), \( E = 0 \).** \( \frac{d}{dt} mv_\parallel = v_\parallel (\frac{B}{B} \cdot \nabla) \frac{B}{B} mv_\parallel = q(\vec{v}_d \wedge \vec{B}) \)
  
  **Curvature drift:** \( \vec{v}_d = \vec{v}_\kappa = -mv_\parallel^2(\kappa \wedge \vec{B})/qB^2 \).

- **Parallel Mirror Force:** \( \frac{d}{dt} mv_\parallel = qE_\parallel - \mu \nabla_\parallel B \)

Total gyrocenter motion is the sum of these drifts.
Intuitive understanding of drifts.
$E \land B$ drift in crossed electric/magnetic fields

Orbit is wider where particle kinetic energy is greater.

Drift velocity $v_E = E \land B / B^2$ same for all particles.

Can also be thought of as a shift to the frame of reference moving with velocity $v_E$ in which $E = 0$.

Any other force $F$ (e.g. gravity) gives similar drift $F \land B / qB^2$. 

Particle Drifts

I H Hutchinson
Polarization drift from non-constant $E$-field

Occurs only when $dE/dt \neq 0$. A “displacement” more than drift.

Example. A finite $E$ suddenly turned on, initially stationary particle.

$$v_p = -\frac{d}{dt} mv_E \wedge B / qB^2 = -m\left( \frac{d}{dt} E \wedge B \right) \wedge B / qB^4 = m\frac{d}{dt} E_\perp / qB^2$$

Displacement $\Delta x = \int v_p dt = m\Delta E_\perp / qB^2$
**\( \nabla B \)** drift from field-strength gradient

Orbit is tighter where \( B \) is bigger. Leads to sideways drift.

Opposite charges gyrate opposite directions; hence opposite drifts. Speed depends on \( v_\perp^2 \) and hence \( \mu \).

Grad-B drift \( v_{\nabla B} = -\mu (\nabla B \wedge \vec{B})/qB^2 \)
Curvature drift from centrifugal force

Definition of curvature
\[ \kappa \equiv \frac{db}{dl} = b \cdot \nabla b \] where \( b = B/B_1 \).

Radius of curvature \( R_c = -\kappa/\kappa^2 \)

Centrifugal force \( F_{cf} = mv^2 R_c/\kappa^2 \).

Resulting curvature drift \( v_\kappa = F_{cf} \wedge B/qB^2 = -mv^2 (\kappa \wedge B)/qB^2 \).
Parallel $\nabla (\mu B)$ mirror force from converging field.

Mirror force arises from the component of the magnetic force in the direction $B(\tilde{x})$ (horizontal here) averaged over a gyro-orbit.

Directed away from stronger-$B$ regions. ($\mu$ is antiparallel to $B$.)
Magnetic Mirror traps some particles
Particles with sufficient $v_\perp$ are reflected from high-B and bounce.
Trapping is determined by the velocity pitch angle

\[ v_{\perp} \]
\[ v_{\parallel} \]

Mirror | Trapped
---|---
\[ \theta_c \]
Loss | Cone

Velocity space at the place where the field is \( B \).

Constancy of \( \mu \) means \( \frac{1}{2}mv_{\perp}^2 = \mu B \). Also \( \frac{1}{2}mv^2 = \text{const} \).

If max field is \( B_m \), particle is trapped only if

\[ \mu B_m = \frac{1}{2}mv_{\perp}^2 \frac{B_m}{B} > \frac{1}{2}m(v_{\parallel}^2 + v_{\perp}^2) \]

i.e. \( \sin^2 \theta > \sin^2 \theta_c = \frac{B}{B_m} \).