RF Heating

General Idea:
Launch a high power wave that is absorbed by the plasma at some appropriate place.

Problem has three coupled parts:
1. Launching: Antenna "coupling" to correct wave(s) in required direction etc.
2. Propagation: Can the wave propagate through the plasma to where it is to be absorbed? Is absorbing region "Accessible"?
3. Absorption: Is wave absorbed? Where does energy go?

All three require understanding of electromagnetic wave behaviour, although at different levels:
1. Vacuum, plasma
2. Plasma propagation
3. Wave damping

Hence, we need to get a good understanding of EM waves in plasmas.
Wave Propagation

Is mostly quite well represented by the Cold Plasma treatment.

In uniform plasma, plane waves satisfy

\[ k \times (k \times E) + \frac{\omega^2}{c^2} E = -i \omega \mu_0 j \]

and \( j = \mathbf{\sigma} \cdot E \). Defining \( \varepsilon = \frac{1}{\varepsilon_0} + \frac{\sigma}{-i \omega \varepsilon_0} \),

\[ k \times (k \times E) + \frac{\omega^2}{c^2} \varepsilon \cdot E = 0. \]

i.e. \( (N N - N^2 + \varepsilon) \cdot E = 0 \) with \( N \equiv \frac{c}{\omega} k \).

Solve the determinant equation \( \|NN-N^2+\varepsilon\| = 0 \),

to arrive at dispersion relation.

Cold Plasma Dielectric Tensor (with \( B \) in \( z \)-direction)

\[ \varepsilon = \begin{bmatrix} \sigma & -iD & 0 \\ iD & \sigma & 0 \\ 0 & 0 & \mathcal{P} \end{bmatrix} \]

\[ S = 1 - \sum_j \frac{\omega_{fi}^2}{\omega - \Omega_j} \]

\[ D = \sum_j \frac{\Omega_j \omega_{fi}^2}{\omega^2 - \Omega_j^2} \]

\[ \mathcal{P} = 1 - \sum_j \frac{\omega_{fi}^2}{\omega_i^2} \]

Since \( \varepsilon \) is indep of \( k \) (\( N \)),

There are generally 2 \( N \)-solutions for given \( \omega \).

Quadratic equation for \( N \) may also have complex solutions implying cut-off or non-propagating regions.

These are the most important for heating accessibility.
Polarization
Each dispersion solution has an accompanying solution for $E$ (of arbitrary total magnitude) direction

Phase velocity is speed at which wave-fronts move
$$v_{ph} = \frac{\omega}{k}$$
in direction $\hat{k}$.

Group velocity is speed at which energy of wave moves
$$v_g = \frac{\partial \omega}{\partial k}$$

Cold plasma $\varepsilon$ is hermitian: $\varepsilon^{xt} = \varepsilon$ implies there is no damping, wave absorption.

Collisions cause damping but in fusion plasmas are practically negligible because

Collision time $\sim 10^{-5} \to 10^{-3} \text{ s} \quad \text{(depend on Te, Ne)}$

Wave transit time $\sim \frac{\text{Size}}{\text{Speed}} \sim \frac{1 \text{ m}}{3 \times 10^8 \text{ m/s}} \sim 3 \times 10^{-9} \text{ s}$.

Wave absorption requires more than the cold plasma physics. But is usually (somewhat) localized, because it requires resonance \( \Delta \omega \). Cut-off is where $N^2 \to 0$. Wave resonance is $N^2 \to \infty$.

For pure propagation there is cut off at $\omega^2 = \omega_p^2$ \( (P=0) \) and (right) $R = \frac{1}{2} (S+D) = 0$. Resonances at lower- and upper hybrid frequencies.
Wave-Particle Resonance.

Alternative dissipation mechanism exists even in a "Collisionless" plasma.
"Resonance" is when a group of particles experiences the wave field in such a way as to be continuously accelerated, instead of being accelerated and then decelerated. This occurs when the frequency of the wave in the particle's rest frame coincides with a resonant frequency of the particle.

This is familiar with composite resonant structures: pipes, strings ...

Resonances of charged particles in B-field are
Cyclotron Frequency $\Omega_j = \frac{q_j B}{m_j}$

Think of particle as a rotating composite but with

```
\begin{align*}
\vec{k} & \quad \vec{k}_i \\
\vec{B} & \quad \vec{B}_i \\
\vec{v}_{\|} & \rightarrow
\end{align*}
```

a steady speed along field. So gyrocenter has velocity $\vec{v}_{\|}$. Phase of wave is

$$\phi = \vec{k} \cdot \vec{x} - \omega t$$

Rate of change of phase at gyrocenter is
\[
\frac{d\phi}{dt} = k \cdot \frac{dx}{dt} - \omega = k_\parallel v_\parallel - \omega
\]

Resonance occurs when this is equal to a harmonic of the gyrofrequency \( \Omega_j \), i.e.

\[
\omega = k_\parallel v_\parallel + n / \Omega_j \quad \text{integer}
\]

Special Case \( n=0 \):

Particle "feels" wave at zero frequency.

Čerenkov or Landau Resonance.

Wave heating a fusion plasma generally requires a Cyclotron or Landau wave-particle resonance for a reasonable proportion of the particle distribution.

Notice that \( \frac{\omega}{k} \) is the phase speed of the wave. So that \( \frac{k_\parallel v_\parallel}{\omega} \sim \frac{v_\parallel}{U_{\text{phase} \parallel}} \). Therefore resonance requires

\begin{align*}
\text{Either} & \quad \text{Slow phase speed} \quad U_{\text{phase}} \sim U_{\parallel(\text{II})} \sim U_{\text{th}} \\
\text{or} & \quad \text{Near Cyclotron Harmonic} \quad \omega \sim n / \Omega_j
\end{align*}

Generally only low cyclotron harmonic resonances, \( n=1 \) or \( 2 \), are strong enough to be useful.

Getting slow phase speed (high \( N \)) requires us to be near a resonance (usually).
Main RF Heating Approaches

1. Ion Cyclotron Range of Frequencies \( \text{ICRF} \)
   Frequency: 20–100 MHz. \( \frac{\Omega_H}{2\pi} = 80 \text{ MHz} \) @ 5.3 T.
   Ion cyclotron damping. (+?)

2. Lower Hybrid wave heating \( \text{LH} \)
   Frequency: 1–8 GHz. \( \frac{\omega_{ph}}{2\pi} = 2 \text{ GHz} \) @ \( 10^{20} \text{ m}^{-3} \)
   Electron Landau damping of slow wave. (+?)

3. Electron Cyclotron Range of Frequencies \( \text{ECRF} \)
   Frequency: 120–240 GHz. \( \frac{\Omega_{ce}}{2\pi} = 140 \text{ GHz} \) @ 5 T.

Other ideas have been pursued; but these are the ones that are established to work.

<table>
<thead>
<tr>
<th>Approach</th>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{ICRF} )</td>
<td>Cheap, easy RF tech.</td>
<td>Launcher close to plasma.</td>
</tr>
<tr>
<td></td>
<td>Local absorption</td>
<td>Impurities generated.</td>
</tr>
<tr>
<td>( \text{LH} )</td>
<td>Current-Drive</td>
<td>Launcher even closer.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Problems penetrating to core.</td>
</tr>
<tr>
<td>( \text{ECRF} )</td>
<td>Remote Launch</td>
<td>High Freq. Sources nonexistent.</td>
</tr>
<tr>
<td></td>
<td>High power density</td>
<td></td>
</tr>
<tr>
<td></td>
<td>V. good, localized absorb.</td>
<td></td>
</tr>
</tbody>
</table>
Landau Damping

A direct calculation of motion in 1-d shows that if the electric field in the direction of motion is $E_\parallel$, with parallel wave-number $k_\parallel$, then power to particles per unit volume is

$$P_{\perp \parallel} = -E_\parallel^2 \frac{\pi q^2 \omega}{2m_k^2} \frac{\partial f}{\partial \omega_\parallel}$$

[There's a similar mechanism called TITMP that replaces the mirror force $-\mu \nabla_\parallel B_\parallel$ for $E_\parallel$.]

Both ions and electrons are capable of absorbing wave energy by Landau Damping.

But electrons move much faster. The wave must be slowed down (from $c$) in order to interact with thermal particles. The amount of slowing required to damp out electrons is less than for ions.

Electron Landau Damping Dominated.
**Cyclotron Resonance.**

Now we turn to a new damping mechanism due to cyclotron resonance
\[ \omega = k_{||} V_{||} + \pi |\Omega|, \quad (n \neq 0). \]

A full calculation of this involves taking the unperturbed orbit:
\[ x(t) = \frac{V_{||}}{ \Omega } \cos \Omega t + \frac{V_{\perp}}{ \Omega } \sin \Omega t + \frac{e}{m} V_{||} t, \]

and calculating \( E(x(t), t) \) due to the wave \( \exp(i(k \cdot x - \omega t)) \). This gives rise to terms like
\[ \exp\left(i\left(k_{||} \cos \Omega t + k_{\perp} V_{||} t - \omega t\right)\right) \quad \text{(letting } k_{\perp} = k_{\perp} \hat{z}) \]

The quantity
\[ \exp\left(i\frac{k_{||} \cos \Omega t}{ \Omega }\right) \]

is periodic in time and can be written as a Fourier series:
\[ \exp\left(i\frac{k_{||} \cos \Omega t}{ \Omega }\right) = \sum_{n=-\infty}^{\infty} i^n e^{\frac{i\pi n}{2}} J_n\left(k_{\perp} V_{||}/\Omega\right) \]

Hence the computation becomes
\[ \sum_{n=-\infty}^{\infty} \exp\left(i\left(-\pi \Omega - k_{\perp} V_{||} - \omega\right) t\right) \cdot J_n\left(k_{\perp} V_{||}/\Omega\right) \]

This shows explicitly the appearance of all the cyclotron harmonics and that those other than \( n=0,1 \) depend on "finite Larmor radius" effects, i.e. the value of \( k_{\perp} V_{||}/\Omega \). For now, let's calculate the particle heating ignoring \( k_{\perp} \) (effectively, \( k_{\perp} V_{||} \) negligible). This gives fundamental second harmonic absorption smaller by \( \sim k_{\perp}^2 V_{||}^2 / \Omega^2 \).
Equation of motion

\[ \dot{\mathcal{U}} = \frac{q}{m} (E + \mathbf{v} \times \mathbf{B}) \]
\[ \frac{1}{\Omega} \dot{U}_x = \frac{E_x}{B} + v_y \]
\[ \frac{1}{\Omega} \dot{U}_y = \frac{E_y}{B} - v_x \]
\[ \Omega = \frac{qB}{m} \quad \text{(has a sign).} \]

Eliminate:

\[ \left[ \frac{1}{\Omega^2} \ddot{U}_x + U_x \right] = \frac{1}{\Omega} \frac{E_x}{B} + \frac{E_y}{B} \]
\[ \left[ \frac{1}{\Omega^2} \ddot{U}_y + U_y \right] = \frac{1}{\Omega} \frac{E_y}{B} - \frac{E_x}{B} \]

Notice that these are driven harmonic oscillators.

Parallel motion is decoupled (provided \( E_z \) is negligible)
so \( U_z = \text{const} \), \( z = U_z t \).

Let the electric field be that of a circularly (left-hand) polarized wave:
\( E_x = E \cos(kz - \omega t) \), \( E_y = E \sin(kz - \omega t) \)

The field at the particle position is (ignoring \( kU_z \))
\( E_x = E \cos(-\omega_0 t) \quad E_y = E \sin(-\omega_0 t) \)

where \( \omega_0 = \omega - kU_z \) is the Doppler shifted frequency experienced by the particle.

Then the driving terms are

\[ \frac{1}{\Omega} \frac{\dot{E}_x}{B} + \frac{E_y}{B} = \left( \frac{\omega_0}{\Omega} + 1 \right) \frac{E}{B} \sin(\omega_0 t) \]
\[ \frac{1}{\Omega} \frac{\dot{E}_y}{B} - \frac{E_x}{B} = \left( \frac{\omega_0}{\Omega} + 1 \right) \frac{E}{B} \cos(\omega_0 t) \]
Notice that if we had chosen $E_y = -E \sin (k_z z - \omega t)$, which is right handed, this is equivalent to changing the sign of $\omega_1$. In particular we would get a multiplicative $(\frac{\omega_1}{\Omega^2} - 1)$. When we obtain the resonant response, it will be significant only near $|\Omega| = \omega_1$. Consequently $(\frac{\omega_1}{\Omega^2} - 1) \approx 0$. This shows that

ions ($\sigma > 0$) are driven only by left-hand waves

electrons ($\sigma < 0$) " right " .

Solution of equations of motion with specified
harmonic drive and initial conditions $\sigma_x = \sigma_y = 0$
is as follows. A particular solution of

$$\frac{1}{\omega^2} \ddot{u} + u = A \sin \omega t$$ is

$$u = \frac{A}{1 - \frac{\omega^2}{\Omega^2}} \sin \omega t$$.

We add to it an amount of the homogeneous
equation's solutions, $\cos \omega t$, $\sin \omega t$ to satisfy the
initial conditions. Hence solution of eqns is ($\sigma_x(0)=0$)

$$\sigma_x = (1 + \frac{\omega_1}{\Omega}) \frac{E}{B} \frac{1}{1 - \frac{\omega^2}{\Omega^2}} \left( \sin(-\omega t) + \sin \Omega t \right)$$

$$\sigma_y = (1 + \frac{\omega_1}{\Omega}) \frac{E}{B} \frac{1}{1 - \frac{\omega^2}{\Omega^2}} \left( -\cos(-\omega t) + \cos \Omega t \right)$$

(And $\sin(-\omega t) + \sin \Omega t = 2 \sin \frac{\Omega - \omega_1}{2} t \cos \frac{\Omega + \omega_1}{2} t$
$-\cos(-\omega t) + \cos \Omega t = 2 \sin \frac{\Omega - \omega_1}{2} t \sin \frac{\Omega + \omega_1}{2} t$)
Resulting perpendicular energy is then
\[ \frac{1}{2} m \mathbf{v}_1^2 = \frac{1}{2} m \left( \frac{(\mathbf{v}_1 \cdot \mathbf{E})}{B} \right)^2 \sin^2 \left( \frac{\Omega - \omega_d}{2} t \right) \frac{4}{(1 - \frac{\omega_d^2}{\Omega^2})^2} = \frac{1}{2} m \left( \frac{E}{B} \right)^2 \frac{4 \sin^2 \left( \frac{\Omega - \omega_d}{2} t \right)}{(1 - \frac{\omega_d^2}{\Omega^2})^2} \]

If the particle were not initially at rest, we should add some extra amount of the homogeneous solution: \( \mathbf{v}_x = \mathbf{v}_o \cos(\Omega t + \phi) \), \( \mathbf{v}_y = -\mathbf{v}_o \sin(\Omega t + \phi) \). This would alter the energy. However, the change in energy \( \frac{1}{2} m (\mathbf{v}_1^2 - \mathbf{v}_o^2) \) would be the above expression plus cross-terms \( \propto \cos(\Omega t + \phi), \sin(\Omega t + \phi) \). The cross-terms average to zero when averaged over phase angles \( \phi \). Therefore, on average, all particles gain energy given above.

Now we must average the energy gain over the parallel distribution function \( f(\mathbf{v}_2) \) to get the total energy gain per unit volume
\[ \Delta E = \int \frac{1}{2} m (\mathbf{v}_1^2 - \mathbf{v}_o^2) f(\mathbf{v}_2) \, d\mathbf{v}_2 = \int \frac{1}{2} m (\mathbf{v}_1^2 - \mathbf{v}_o^2) f(\mathbf{v}_2) (-\frac{d\omega_d}{k}) \]
\[ = \frac{1}{2} m \left( \frac{E}{B} \right)^2 \int \frac{4}{(1 - \frac{\omega_d^2}{\Omega^2})^2} \sin^2 \left( \frac{\Omega - \omega_d}{2} t \right) f(\mathbf{v}_2) \, d\omega_d \]

Now recognize that as \( t \) gets larger, \( \frac{\sin^2 \left( \frac{\Omega - \omega_d}{2} t \right)}{(1 - \frac{\omega_d^2}{\Omega^2})^2} \) becomes a narrow localized function near \( \omega_d = \Omega \).
Hence, in this limit,

\[ \Delta \varepsilon = \frac{1}{2} m \left( \frac{E}{b} \right)^2 \cdot \delta f(\omega) \int \frac{\sin^2 \frac{\Omega - \omega t}{2}}{(\Omega - \omega)^2} \left( \frac{-d\omega}{k} \right) \]

\[ = m \left( \frac{E}{b} \right)^2 \Omega^2 f\left( \frac{\omega - \Omega}{k} \right) \frac{\pi}{k} \int \frac{\sin^2 x}{x^2} \, dx \]

\[ = \frac{E^2 q^2}{m} \frac{\pi}{k} f\left( \frac{\omega - \Omega}{k} \right) t. \]

Consequently the heating power density is

\[ P = \frac{E^2 q^2}{m} \frac{\pi}{k} f\left( \frac{\omega - \Omega}{k} \right). \]

Note that \( k \) here is the parallel wave number.

This expression is highly reminiscent of the corresponding power for Landau damping, viz:

\[ P = -\frac{E^2 q^2}{2} \frac{\pi}{m} \frac{\omega}{k} \left( \frac{\partial f}{\partial \nu} \right)_{\Omega \nu}. \]

However, for cyclotron absorption the polarization is key. I.e. what fraction of wave is left-hand circular polarized (for ion damping). [Right: Electrons]

One can immediately show from the cold plasma dispersion tensor that when \( \omega \rightarrow \Omega j \), \( E_y \rightarrow i \frac{\omega}{k} E_x \)

i.e. wave polarization is exactly wrong for damping at the fundamental resonance in cold plasma limit.
Summary: Collisionless Damping/Heating

Requires wave-particle resonance
\[ \omega = k_{\parallel} v_{\parallel} + n \Omega \]

\( n=0 \): Landau
\( n\geq 1 \): Cyclotron

Power absorbed per unit volume

\( n=0 \)
\[ P_{LD} = -\left| E_{\parallel} \right|^2 \frac{\pi}{2} \frac{q^2}{mk_{\parallel}} \left( \frac{\omega}{k_{\parallel}} \right) \frac{\partial f_{\parallel}}{\partial \Omega} \left| \frac{\omega}{k_{\parallel}} \right| \frac{\omega}{k_{\parallel}} \exp \left( -\frac{m\omega^2}{2k_{\parallel}^2} \right) \]

\[ P_{\text{Maxwell}} = -\frac{B_{\perp}^2}{B_0^2} 2 \frac{\pi m}{4} \omega \int_0^\infty \frac{v_{\perp}^2}{\omega_{\parallel}} \frac{\partial f_{\perp}}{\partial v_{\perp}} 2\pi v_{\perp} dv_{\perp} \]

\[ = \frac{B_{\perp}^2}{B_0^2} \frac{\pi m}{4} \omega \left( \frac{n}{2\pi} \right)^n \frac{\omega}{k_{\parallel}} \sqrt{\frac{T}{m}} \exp \left( -\frac{m\omega^2}{2k_{\parallel}^2} \right) \quad \text{(Maxwellian)} \]

\( n=1 \)
\[ P_{\Omega} = \frac{E_{L}}{2} \frac{q^2}{m} \frac{\pi}{k_{\parallel}} \frac{\partial f_{\parallel}}{\partial \Omega} \left( \frac{\omega - \Omega}{k_{\parallel}} \right) \]

\[ = \frac{E_{L}}{2} \frac{q^2}{m} \frac{\pi}{k_{\parallel}} \left( \frac{m}{2\pi T_{\parallel}} \right)^{\frac{1}{2}} n \exp \left( -\frac{m(\omega - \Omega)}{2k_{\parallel}^2} \right) \quad \text{Left: ions, Right: Electrons} \]

\( n=2 \)
\[ P_{2\Omega} \sim (k_{\parallel} v_{\perp})^2 \quad \text{smaller (at 2\Omega resonance)} \]

\[ \omega_i = -\frac{P}{2W} \quad \text{for real } k, \text{ or else}: \]
\[ k_i = +\frac{P}{2 \left| \left| E \times B \right| \right|} \quad \text{for real } \omega. \quad \text{Spatial attenuation} \]

Need to calculate wave polarization.
Electron Cyclotron Heating

Two harmonics (1, 2) have sufficient damping to absorb wave in 1 pass through plasma.

Accessibility
Can we get the wave to the resonant layer? Cut-offs & Resonances. (Perp Prop)

$N^2 = \frac{RL}{S}$ (x mode) Resonance: $\omega_{\text{un}}^2 = \omega_{pe}^2 + \omega_{ce}^2$

Cut-offs $R = 0$ right-hand $R \approx 1 - \frac{\omega_{pe}^2}{\omega(\omega + \Omega_e)}$

$\Rightarrow \omega_r = \frac{1}{2} \left\{ |\Omega_e| + \sqrt{|\Omega_e|^2 + 4\omega_{pe}^2} \right\}$

This is somewhat above the first cyclotron harmonic by an amount that increases with $N_e$ ($\propto \omega_{pe}^2$).

$L = 0$ left-hand $L \approx 1 - \frac{\omega_{pe}}{\omega(\omega + \Omega_e)}$

$\omega_L = \frac{1}{2} \left\{ -|\Omega_e| + \sqrt{|\Omega_e|^2 + 4\omega_{pe}^2} \right\}$ below $|\Omega_e|$. 

$\rightarrow 0$ as $N_e \rightarrow 0$. 
**Summary**

1. $x$-mode at $1\Omega$ accessible only from inboard, not outboard.
2. $x$-mode at $2\Omega$ outboard accessible if $\omega_{ro} < 2\Omega$ which is $\omega_p^2 < 2\Omega^2$.
3. $o$-mode at $1\Omega$ accessible if $\omega_{pe} < \Omega$ (o-mode damping at $2\Omega$ too weak).
In terms of wave (source) frequency:

0-mode, $1\Omega$ : $\omega_p^2 < \omega^2$

x-mode, $2\Omega$ : $\omega_p^2 < \frac{\omega^2}{2}$

The upper density limit is lower for $2\Omega$ if $B$ is regarded as a free choice. (Not usually true).

Typical $5T$ : $\frac{\omega}{2\pi} = 140$ GHz

$\frac{2\omega}{2\pi} = 280$ GHz.

Sources with respectable power (~1 MW) do not exist above about 100 GHz but are under development. Gyrotron technology would need many advances to give high power for long pulse at 280 GHz. This is the present limiting factor for usefulness of ECH.

Absorption: Single pass transmission = $\exp(-\tau)$

$1\Omega$, 0-mode $\tau \approx \frac{R}{\varepsilon} \frac{\pi}{2c} \omega_p^2 \left(1 - \frac{\omega_p^2}{\omega^2}\right) \left(\frac{T_e}{m_e c^2}\right)$

$x$-mode $\tau \approx \frac{R}{2\Omega} \frac{\pi}{2c} \omega_p^2 \left(4\frac{T_e}{m_e c^2}\right)$

Almost always $\tau \gg 1 \Rightarrow$ Complete single pass absorption.
Propagation in Non Uniform Plasmas.

When plasma non-uniformly (inhomogeneity) is strong, Plane-wave approximations may be inadequate. Even if the geometric optics approach is valid, any symmetries the plasma has help to establish the way propagation occurs. Example: translational symmetry:

\[
\text{Plasma Indep. of } z \rightarrow
\]

A wave that starts as \( \propto \exp(i k_z z) \) stays purely that. No coupling to other \( k_z \)'s. (Snell's Law)

Tokamak Indep of \( \phi \): almost same. Actually it is '\( n \)' that is conserved \( \exp(i \phi) = \exp(i \frac{n}{R} \Phi) \sim \exp(i k_z z) \) (\( k_z R \) is conserved). This is essentially exact.

Approximately: \( k_z \) is nearly \( k_{\parallel} \) since \( B_z \gg B_{\parallel} \). This helps to understand propagation and accessibility. Separate out the \( k_z (N_{\perp}) \) (\( k_{\parallel}, N_{\parallel} \)) part of the wave's variation, and regard that as fixed by the launching configuration.
Dispersion Tensor in terms of $N_{||}$, $N_{\perp}$

$$
\mathbf{D} = \begin{bmatrix}
-N_{||}^2 + S & -iD & N_{||} N_{\perp} \\
+iD & -N_{||}^2 N_{\perp}^2 + S & 0 \\
N_{||} N_{\perp} & 0 & -N_{\perp}^2 + P
\end{bmatrix}
$$

Regard $N_{||}$ as given; solve $\|\mathbf{D}\| = 0$ for $N_{\perp}$.

Results in:

$$
\|\mathbf{D}\| = a N_{\perp}^4 - b N_{\perp}^2 + c = 0
$$

with

$$
a = S, \quad b = (S-N_{||}^2)(S+P)-D^2, \quad c = P[(S-N_{||}^2)^2-D^2]
$$

solution

$$
N_{\perp}^2 = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2c}{b + \sqrt{b^2-4ac}}
$$

Two solutions for $N_{\perp}^2$ are called, generically

"Slow Wave", "Fast Wave": higher $N_{\perp}$, lower $N_{\perp}$.

In various areas of parameter space these are sometimes called other things, e.g.

**Lower Hybrid wave** → **Slow Wave**
**Magnetosonic wave** → **Fast Wave**.

Slow wave experiences a resonance at $a = S = 0$.

$$
S = 1 + \frac{\omega_{pe}^2}{\Omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2} = 0 \quad \text{[Lower Hybrid Resonance]}
$$

Fast waves have cut-off at $c = 0$.

Slow waves, Fast waves, & [Slow $P = 0$]

Fast $[S-N_{||}^2] = 0$.

Fast and slow branches merge at $b^2 - 4ac = 0$. 

Lower Hybrid Wave  (Slow wave in general vicinity of LH resonance).

Assume \( \omega_{pe} \sim |\Omega_e| \) then

\[
S \approx 1 + \frac{\omega_{pe}^2}{\Omega_e^2} - \frac{\omega_i^2}{\omega^2} \quad \sim 1
\]

\[
D \approx \frac{|\Omega_e| \omega_{pe}^2}{\omega} \Omega_e^2 \quad \sim \left( \frac{m_i}{m_e} \right)^n
\]

\[
P \approx 1 - \frac{\omega_{pe}^2}{\omega^2} \quad \sim \frac{m_i}{m_e}
\]

Think about a stratified plasma such that \( \omega_{pe}^2 \propto n_e \) varies from zero (at edge) to some value, but \( \Omega_e \) and \( \omega \) are constants. Then wave parameters vary thus:

(assuming \( \frac{\Omega_e^2}{\omega^2} > \frac{\omega_{pe}^2}{\omega_i^2} \)

i.e. \( \omega < |\Omega_e \Omega_i| \),

so that a LH resonance exists.)

There are then two possible topologies of the refractive index depending whether or not there exists a place where \( b^2 + 4ac = 0 \) (goes negative) where it does, \( N_i^2 \) becomes complex: wave is evanescent and cannot propagate.
Case 1 \( N_{\parallel} > N_{\parallel c} \).

Case 2 \( N_{\parallel} < N_{\parallel c} \).

![Graphs showing two cases: Case 1 with \( N_{\parallel} > N_{\parallel c} \) and Case 2 with \( N_{\parallel} < N_{\parallel c} \).]

Criterion distinguishing the two cases can be written (approx cold plasma) \( N_{\parallel}^2 \leq N_{\parallel c}^2 \)  where

\[
N_{\parallel c}^2 = 1 + \frac{\Omega_{pe}^2}{\Omega_e^2} \quad \text{(evaluated at } \Omega_{\parallel})
\]

Case 2 does not allow wave to penetrate to plasma center. Accessibility requires \( N_{\parallel} > 1 \) (slow in parallel direction as well as perp.). Typically \( N_{\parallel} \approx 1.4 \) (e.g. \( \Omega_{pe} = 10\Omega_e \)).

Thermal corrections may be important sometimes, both for \( N_{\parallel c} \) and \( \Omega_{\parallel} \).
Damping Mechanisms for LH waves.

Ion Damping: (Perpendicular) Ion Landau Damping

since \( \omega \gg |\Omega_i| \) . Needs (ion) Landau resonance

\[ \omega - k u_i = 0 \]

\( k \) is mostly \( k \parallel \). Need thermal ions to satisfy this
so need \( k_\perp \sim \frac{\omega}{u_i} \), i.e. \( N_\perp \sim \frac{c}{u_i} \).

Occurs only close to resonance.

Have to choose field/density such that \( \omega = \omega_{LH} \) at the place required to be heated, usually the center

\[ \frac{1}{\omega^2} = \frac{1}{\omega_{pi}^2} + \frac{1}{\omega_{ei}^2} \]

For given \( B \& \omega \), raising \( N_e \) moves resonance outward.

Ion effects give maximum density for electron heating (below).

Non-linear effects can also occur that tend to obscure what is happening & make heating less efficient.
Electron Landau Damping (parallel, $\omega \ll \Omega_e$)

$$\omega - k_{||} U_{Te} = 0 : \quad k_{||} \approx \frac{\omega}{U_{Te}} , \quad N_{||} \sim \frac{e}{U_{Te}} .$$

Is so strong that can take place on the tail of the distribution: $U_{||} = 3 \rightarrow 5$ $U_{Te}$ has enough electrons to damp very quickly.

Electron Temperature determines damping of given $N_{||}$.

In a rising $T_e$, wave penetrates till it reaches place where

$$\frac{e}{N_{||}} \sim 5 \, U_{Te}$$

then damps out. At this point:

$$T_e \sim \frac{1}{5^2} \frac{m_e c^2}{N_{||}^2} \sim \frac{20 \text{ kV}}{N_{||}^2}$$

We must choose $N_{||}$ so that damping occurs in the right place.

A problem: Accessibility:

$$N_{||}^2 > 1 + \frac{\omega_{pe}^2}{\Omega_e^2}$$

Damping:

$$N_{||}^2 \leq \frac{20 \text{ kV}}{T_e}$$

Are these compatible? Not really, at high $T_e$.

In a high $T_e$ machine LH waves tend to be too strongly absorbed near the edge. Heating (or CD) may not get to the center.
RF Current-Drive

LH waves are by far the most successful RF current-drive scheme. Megamp-level drive has been obtained. TRIAM-IM has run LH current-driven plasmas of duration over 1 hour.

Mechanism: Relies on asymmetric wave spectrum interacting with electrons on tail of distribution.

Landau Damping flattens the distribution function. Also, hotter electrons slow down (being dragged back into the bulk) less rapidly than colder.
Efficiency of Current Drive

Expressed in terms of a figure of merit:
\[ \eta = \frac{(n/10^{20} \text{m}^{-3})(I_p/MA)R}{(P/MW)} \leq \text{absorbed power} \]

Physics behind this:

1. Current carrying electrons have collision frequency with background ions \( \propto n \)
2. Larger major radius \( \Rightarrow \) more cc electrons to collide
3. \( I_{\text{plasma}}/P_{\text{power}} \) says how efficient the scheme is.

Value of \( \eta \) depends on the energy (and hence collisionality) of cc electrons. Higher energy is better but when the electrons become relativistic their velocity no longer increases so collisionality has a lower limit. Theoretical maximum \( \eta \sim 1 \).

Experimentally LHCD can get \( \eta \sim 0.2 \) alone and perhaps \( \eta \sim 0.5 \) with synergistic effects from ICRF (JET).

Note that even with \( \eta \sim 1 \), ITER would need \( \sim 150 \text{ MW} \) to drive all current. Bootstrap “fraction” may be enough to make steady reactor feasible.
Wave Launching

How do you launch a wave with $N_{||}^2 > 1$?

A propagating wave (planar) always has

$$k^2 = k_{||}^2 + k_\perp^2$$

Hence $k_{||}^2 < k^2$ i.e. $N_{||}^2 < N^2$ but in free space $N^2 = 1 \Rightarrow N_{||}^2 < 1$. No good.

Answer: wave is non-propagating in $\perp$ direction, in free space $N_{\perp}^2 < 0$ allows $N_{||}^2 > 1$. Wave is evanescent outside the plasma.

Method of exciting such waves: Stacked Waveguides:

E-field $\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow 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Because wave is evanescent outside plasma (precisely: for \( \omega^2 < \omega_p^2 \)) we can't have the launcher too far from plasma.

Example: \( \frac{\omega}{2\pi} = 4.6 \text{ GHz} \). Wavelength \( \sim 6 \text{ cm} \).

We need to be a fair bit closer to plasma than this, for good coupling. Typically \( \sim 1 \text{ cm} \) \((=|k|; x| \approx 1)\).

Lower Hybrid "Grill" needs to be close to plasma, a disadvantage.

Technology: Waveguides.

Windows.

Generator: Klystron (electron tube).

Broadly: challenging but feasible. Multi-megawatt systems in operation. E.g. JET \( \sim 8 \text{ MW} \).
Grill IV

main waveguide -----> 3 sub-waveguides
3-div. x 8-main = 24 waveguides in toroidal direction

Grill V

main waveguide -----> 12 sub-waveguides
12-div. x 4-main = 48 waveguides in toroidal direction

JT-60U LH Grills
Ion Cyclotron Heating

Is very complicated because
(1) Wavelengths are comparable to plasma size.
So geometric optics is a poor approx.
(2) Fundamental absorption is weak because of
polarization effects.
(3) Mode conversion is an important process
(and we are not getting into it).

Our approach is highly simplified.
Cold Plasma dispersion relation is approximated
by noting that $P \sim \frac{\omega_e^2}{\omega_P^2}$ is large. So to make
the dispersion determinant zero without using a
resonance in the other components to cancel $P$
we need the cofactor approximately zero:

$$
\begin{bmatrix}
-N_{||}^2 + S & -iD \\
+ iD & -N_{||}^2 + S
\end{bmatrix}
\begin{bmatrix}
E_x \\
E_y
\end{bmatrix} = 0
$$

Physics behind this approx is that $E_z$ is "shorted
out" by electrons flowing along $B$.
Solve this for $N_{\perp}$:

$$
N_{\perp}^2 = \frac{(S-N_{||}^2)^2 - D^2}{S-N_{||}^2} = \frac{(R-N_{||}^2)(L-N_{||}^2)}{S-N_{||}^2}
$$

This is just Fast Wave (slow lost in approx.).
Polarization at fundamental ion resonance $\omega \rightarrow \Omega_i$

$S \rightarrow -\frac{\omega_i^2}{\omega^2 - \Omega_i^2}$  ;  $D \rightarrow \frac{\Omega_i}{\omega} \frac{\omega_i^2}{\omega^2 - \Omega_i^2}$ (both $\rightarrow \infty$)

from top line of dispersion equation , $D \cdot E = 0$ , get

$$(-N_{ii}^2 + S) E_x + iD E_y = 0$$

$$\Rightarrow \frac{E_x}{E_y} = \frac{-iD}{S - N_{ii}^2 + S} \rightarrow \frac{+iD}{S} \rightarrow -i \frac{\Omega_i}{\omega} \rightarrow -i$$

all as $\omega \rightarrow \Omega_i$ ($\Omega_i$ is +ve)

This is an exactly right-hand circular polarized wave. It is orthogonal to the required left-hand component that causes cyclotron heating.

Cold plasma treatment shows fast-wave polarization wrong: "no" cyclotron damping at fundamental. (Basically right but oversimplified).

Options

(A) Use harmonic (usually 2nd). Damping smaller by $\sim \frac{(k_i v_e)^2 \Omega_i}{\Omega_i^2}$. Tends preferentially to heat ions on tail of distribution.

(B) Mix in small amount of a different ion. Hope polarization is dominated by majority but minority damping enough. (It works)
Actually, cold plasma treatment of wave polarization would predict the same problem at its resonance because the minority contribution to $E$ will dominate at its resonance. However, the simple picture that minority can be ignored in polarization turns out to be OK in many situations with low ($\leq 5\%$) minority concentration.

Simplified dispersion relation has

**Wave Resonance (not wave-particle)** at $N_{||}^2 = S$

**Cut-off** at $N_{||}^2 = R$ and $N_{||}^2 = L$.

Resulting regions for light minority e.g. $H$ in D-maj.

(Mode conversion to Ion Bernstein wave not discussed.)

Resonance & cut off (ion-ion hybrid) arise because of cancellation between two large ion terms in $S$ & $R, L$.

Approximate solution, just keeping those large terms:

$S = N_{||}^2 \text{Resonance: } \omega^2 = \frac{\omega_i^2 \Omega_z^2 + \omega_p^2 \Omega_1^2}{\omega_i^2 + \omega_p^2}$

$L = N_{||}^2 \text{ cut-off: } \omega = \frac{\omega_i \Omega_z + \omega_p \Omega_1}{\omega_i^2 + \omega_p^2} = \omega_-$
Recall \( W_{pj} \propto \frac{\varepsilon_j n_j}{m_j} \).

If \( \Omega \propto B \propto \frac{1}{R} \), distance between cyclotron resonance for minority (2) and cut-off is easily calculated:

Cyclotron Resonance: \( \omega = \Omega_2(R) \), \( R = R_0 \), say.

\( L = N_i \) cut-off: \( \omega = \frac{\omega_{p1}^2 \Omega_2 + \omega_{p2}^2 \Omega_1}{\omega_{p1}^2 + \omega_{p2}^2} \)

Solve for \( \Omega_2(R_c) = \omega \left( \frac{\omega_{p1}^2 + \omega_{p2}^2}{\omega_{p1}^2} \right) - \frac{\omega_{p1}^2}{\omega_{p1}^2} \Omega_1(R_c) \)

\[ \frac{\Omega_2(R_c)}{\Omega_2(R_c)} = \frac{R_c}{R_0} = \left( 1 + \frac{\omega_{p2}^2}{\omega_{p1}^2} \right) \frac{\omega_{p1}^2}{\omega_{p1}^2 + \omega_{p2}^2} \]

\[ \text{So} \quad \frac{R_0 - R_c}{R_0} = \frac{\omega_{p2}^2 \left( 1 - \frac{\Omega_1}{\Omega_2} \right)}{\omega_{p1}^2 + \omega_{p2}^2} = \frac{n_i q_{12}^2 m_2}{m_1 n_i^2/m_1 + n_i q_{12}^2/m_2} \left( 1 - \frac{q_1 m_1}{q_2 m_2} \right) \]

For small \( n_2 \), \( R_0 - R_c \propto n_2 \). E.g. \( n_2 = 0.05 n_1 \), \( \psi = q_{12} \)

\( m_2 = \frac{m_1}{2} \):

\[ \frac{R_0 - R_c}{R_0} = 0.045 \quad \text{~cm in \( R=1 \mu \) tokamak} \]

This defines a characteristic radial width of the region where minority ion effects dominate the wave propagation.

Wave absorption also takes place with a characteristic width. Recall that absorption \( \propto f \left( \frac{\omega - \omega_i}{k_{ni}} \right) \).

For given \( \omega \) there is a range of \( \Omega_2 \) for which this is \( \pm 0 \).
Cyclotron resonance width \( \Delta \Omega_i \sim k_{\|} U_{ei} \)

\[ \Rightarrow \Delta \sigma = R_0 \frac{\Delta \Omega_i}{\Omega_i} \sim R_0 \frac{k_{\|} U_{ei}}{\Omega_i} \quad (i=2\text{, minority}) \]

For reasons too complicated to discuss, the minority can be treated as having only unimportant effects on propagation if \( \Delta > R_0 - R_c \) i.e.

\[
\frac{n_z}{n_i \left( \frac{q_i m_i}{q_z m_z} \right)} < \frac{k_{\|} U_{ez}}{\Omega_z}
\]

i.e. for small enough minority fraction. (Typically <0.05)

Then absorption is calculated using the polarization of wave ignoring the minority species. The fraction of left-hand polarization can be shown to be

\[
\left| \frac{E_L}{E_y} \right|^2 = \left[ 1 - \frac{\Omega}{\Omega_i} - \frac{N^2_z}{s} \right]^2 = 1 \text{ at } 2\Omega_1 \text{ (Heid)},
\]

\[
\text{and } \approx \frac{1}{4} \text{ at } \frac{4}{3} \Omega_1 \text{ (He}^3\text{inD)}
\]

\( \left( \frac{N^2_z}{s} \text{ usually negligible} \right) \)

In this case, the damping leads to a complex \( k_1 \), for real \( \omega \). Damping in space. At any point in the resonance there is a certain imaginary part \( k_i \).

Total transmission is obtained by using

\[ E \propto \exp i(k_{\|} x - \omega t) \Rightarrow \text{Power transmitted is fraction } \exp(-2\eta) \text{ where } \eta = \int k_i(x) \, dx \]
Decomposition into different polarization components

We generally write a wave as being given by

$$\mathbf{E} = \Re(\mathbf{E} e^{i\phi})$$

in a coordinate system x-y in the relevant plane.

The $\Re$ reminds us that this is the real electric field, while $\mathbf{E}$ is complex.

Any polarization is described by $\mathbf{E}$ and can be expressed in terms of any two independent polarizations. Generally they are taken to be given by orthonormal basis vectors. The two natural choices are

1) Linear $\mathbf{e}_x = (1, 0)$  $\mathbf{e}_y = (0, 1)$
2) Circular $\mathbf{e}_x = \frac{1}{\sqrt{2}}(1, i)$  $\mathbf{e}_y = \frac{1}{\sqrt{2}}(1, -i)$

[Note: orthonormality means $\mathbf{e}_x \cdot \mathbf{e}_x^* = 1, \delta_{\alpha\beta}$]

Any general field can be written

$$\mathbf{E} = (\mathbf{e}_x^* \cdot \mathbf{E}) \mathbf{e}_x + (\mathbf{e}_y^* \cdot \mathbf{E}) \mathbf{e}_y$$

$$= (\mathbf{e}_x^* \cdot \mathbf{E}) \mathbf{e}_x + (\mathbf{e}_y^* \cdot \mathbf{E}) \mathbf{e}_y$$

It is then natural to call these coefficients, e.g.

$$\mathbf{e}_x^* \cdot \mathbf{E} = E_x, \quad \mathbf{e}_y^* \cdot \mathbf{E} = E_y$$

However this notation is NOT what is used in the notes; this may cause some confusion.
In deriving the power absorption we considered a field \( \mathbf{E}_x = E' \cos \phi \), \( \mathbf{E}_y = E' \sin \phi \), which, in terms of complex notation is equivalent to
\[
E = (1, -i) E'
\]
[Actually the notes don't use the prime, but I am trying here to be completely clear].

This choice is convenient and means we don't have lots of \( \sqrt{2} \) factors. However the magnitude of this electric field is actually \( E \cdot E^* = 2 E'^2 \).

After deriving the power absorption in terms of \( E' \), \( E' \) is subsequently called \( E_L \) because it is a left hand wave. [This is where the notation becomes inconsistent with the normalization and the other notation \( E_L = \varepsilon_L^* \cdot E \)]. In fact,
\[
\varepsilon_L^* \cdot E' = \sqrt{2} E'
\]

so there is an extra factor of \( \sqrt{2} \). Therefore, whenever \( E_L \) appears in the notes referring to the wave that is ion cyclotron damped, it really means \( E' = \sqrt{2} \varepsilon_L^* \cdot E \).

**In Summary**

When using \( E_L \) in notes:

\[
E_L = \sqrt{2} \varepsilon_L^* \cdot E = (1, i) E = E_x + i E_y.
\]

[Then \( |E_x|/|E_y| = 1 \) implies \( E_x = 0 \), linearly polarized].

More conventionally
\[
\varepsilon_L^* \cdot E = \frac{1}{\sqrt{2}} (1, i) E = \frac{1}{\sqrt{2}} (E_x + i E_y)
\]
is usually referred to as \( E_L \).

[Then \( |E_x|/|E_y| = 1 \) implies fully LH circular polarization].
Relationship between absorption and $k_i$.

If radiative (wave) power/unit area is $S$ (Poynting flux), and power absorption per unit volume is $P$, then
\[
\frac{dS}{dl} = -P \propto -|S|
\]
and generally $|S| \propto E^2$ so $E^2$ falls off with exponential inverse length $2k_i$, such that
\[
2k_i = \frac{|P|}{|S|}.
\]

Now Poynting Flux is $E \cdot H$ (instantaneously). Averaged over a period,
\[
\bar{S} = \frac{1}{2} \mathcal{F}(E \cdot H^*) = \frac{1}{2\mu_0} \mathcal{F}(E \cdot B^*)
\]
Near perpendicular propagation ($k_z = k_x \gg k_y$) use
\[
\nabla \cdot E = -\frac{\partial B}{\partial t} \quad \Rightarrow \quad i k_z E_y = -i \omega B_z
\]
So $B_z = -\frac{N_x}{c} E_y$. Substitute:
\[
\bar{S} = \frac{N_x}{2\mu_0} |E_y|^2 \hat{z} = \frac{cN_x}{2} \varepsilon_0 |E_y|^2 \hat{z}
\]
Hence $2k_i = \frac{|P|}{\frac{cN_x}{2} \varepsilon_0 |E_y|^2}$ (near perf.)

Use previous $P$:
\[
\left(\frac{P}{|S|}\right) 2k_i = \left(\frac{|E_L|^2}{|E_y|^2}\right) \frac{q^2}{m_2} \frac{\pi}{k_{\parallel}} \int \frac{f\left(\omega - 2\pi\right)}{k_{\parallel}} d\omega
\]
Minority Absorption.
Hence
\[ 2\eta = \int \frac{P}{L} \, dl = \frac{|E_L|^2}{|E_y|^2} \frac{q_2^2}{m_e c N_\perp} \int f_{2\perp} \left( \frac{\omega - \Omega_2}{k_\parallel} \right) \, dR \]

But, since \( \Omega_2 = \Omega_2(R_0) \frac{R_0}{R} \approx \Omega_2(R_0) \left(1 - \frac{R - R_0}{R_0}\right) \)
\[ \int f_{2\perp} \left( \frac{\omega - \Omega_2}{k_\parallel} \right) \, dR \approx \int f_{2\perp} \left( \frac{\omega}{k_\parallel} \right) \, d(\Delta R) = \frac{R_0 k_\parallel}{\omega} \int f_{2\perp} \, d\nu = \frac{R_0 k_\parallel}{\omega} \mathcal{N}_2 \]

Hence
\[ 2\eta = \frac{|E_L|^2}{|E_y|^2} \frac{\omega_{p2}^2}{\omega c N_\perp} \]

Absorption: \( 1 - e^{-2\eta} \)
Transmission: \( e^{-2\eta} \)

Value of \( N_\perp \), ignoring \( N_\parallel \) in the dispersion relation and retaining only the majority ion terms in \( S, D \) etc is
\[ N_\perp \approx \frac{R L}{s} \approx \frac{\omega_{p1}^2}{\omega^2} \]

Substitute to get:
\[ 2\eta = \frac{|E_L|^2}{|E_y|^2} \frac{\omega_{p1}}{c N_\perp \left( \frac{q_2^2 m_i}{q_1^2 m_1} \right)} R_0 \]

\[ \frac{|E_L|^2}{|E_y|^2} = \frac{1}{|q|} \int H^1 \text{ d}H^3 \] \( n_2 \) minority density \( R_0 \) major radius

Valid when \( \frac{n_2}{n_1} \ll \frac{k_\parallel U_{Te2}}{\Omega_2} \)

If this inequality is not satisfied, then \( 2\eta \) is approximately multiplied by \( \left( \frac{n_2}{n_1} k_\parallel U_{Te2} \right)^2 \) (and we are in the mode conversion regime)

so we could write
\[ 2\eta \approx \frac{|E_L|^2}{|E_y|^2} \frac{\omega_{p1}}{c N_\perp \left( \frac{q_2^2 m_i}{q_1^2 m_1} \right)} R_0 \frac{1}{1 + \left( \frac{n_2}{n_1} \frac{\Omega_2}{k_\parallel U_{Te2}} \right)^2} \]
Example: Alcator C-Mod.

\[ \frac{\Omega}{2\pi} = 80 \text{ MHz} \quad R_0 = 0.67 \text{ m} \]

\( k_{\parallel} \) determined by antenna
2 strips 180° out of phase
about 0.25 m apart

\[ k_{\parallel} d = \pi \]

\[ k_{\parallel} = \frac{\pi}{d} \times \frac{\pi}{0.25} \text{ m}^{-1} \]

If \( T_{e2} \) is 2 keV then

\[ \frac{k_{\parallel} T_{e2}}{\Omega} = \frac{\pi^2 \times 10^{-16}}{2\pi \times 80 \times 10^6} = 10^{-2} \]

Very low minority density is required to get into valid absorption regime (VSW mode conversion). Actually the minority will get a lot hotter because of power going into it. Even so we want \( \frac{n_{H}}{n_{D}} \sim 1-3\% \).

Suppose \( n_{D} = 10^{20} \text{ m}^{-3} \) ⇒ \( \omega_{pe} \sim 10^{10} \text{ s}^{-1} \)

\[ 2\eta \approx 1 \cdot \frac{10^{10}}{3 \times 10^8} 0.01 (2) 0.67 \approx 0.5 \]

Hence fraction absorbed in a single pass is approximately \( 1 - e^{-0.5} = 0.4 \) (40%).

Actual heating efficiency is better than this, close to 100%, because power bounces round till absorbed.
ICRF Launching

Because $N_{||} > 1$ of waves we want, they are evanescent at plasma edge and outside. In fact, the exponentiation coefficient is close to the same as $k_{||}$ because $N_{||} \approx 5$ so wave goes like $\exp(ik_{||}z - k_{||}x)$. Good coupling therefore requires antenna to be quite close to the plasma: $\lesssim \frac{1}{k_{||}}$

Antenna: Poloidal Loop:
- Gives $E_y$, $B_z$ polarization as required.

"Monopole" single strap gives rise to $k_{||}$ spectrum:
- A lot of power is at $k_{||} \approx 0$.
- More mode conversion, worse damping.

"Dipole" Antenna two straps gives rise to spectrum
Alcator C-Mod.

(a) single strap antenna

(b) two-strap antenna (out of phase)

Radiated spectrum $n_\parallel = \frac{c k_\parallel}{\omega}$
FISIC \( n = 5 \)

\[ \text{D(H) scenario wave fields (full wave code)} \]
Impurity Generation. Avoidance is important.
Dipole helps to minimise RF sheath effects.
Choose wall/antenna materials for low sputtering.
Coat antenna components with e.g. B$_4$C.

Edge Plasma Interaction. (Arcs, parasitic coupling)
Minimize by using "Faraday Screen" in front of antenna strap. Slots prevent EM fields from being shorted out.

Electrical Optimization.
Match transmitter power to antenna using resonant circuit & tuners (coax technology).
High voltage. Vacuum feed-throughs.

Forces (Disruptions etc)
Make antenna, box strong (e.g. inconel) CU plate.

Thermal Effects
Prevent direct plasma contact to screen by limiter.
Passive cooling: radiation, conduction.
Active cooling for long pulse.
**Electrostatic Approximation** for (Plasma) Waves.

The dispersion relation is written generally as

\[ N \wedge (N \wedge E) + \varepsilon \cdot E = N(N, E) - N^2 E + \varepsilon \cdot E = 0 \]

Consider \( E \) to be expressible as longitudinal and transverse components \( E_L, E_T \) such that \( N \wedge E_L = 0 \), \( N \cdot E_T = 0 \). Then the dispersion relation can be written

\[ N \cdot E_L - N^2 (E_L + E_T) + \varepsilon \cdot (E_L + E_T) \]

\[ = -N^2 E_T + \varepsilon \cdot E_T + \varepsilon \cdot E_L = 0 \]

or

\[ (N^2 - \varepsilon) \cdot E_T = \varepsilon \cdot E_L \]

Now the electric field can always be written as the sum of a curl-free component plus a divergenceless component, e.g. conventionally

\[ E = -\nabla \phi + \mathbf{A} \]

Curl-free, Divergence-Free, Electrostatic Electromagnetic

and these may be termed electrostatic and electromagnetic parts of the field.
For a plane wave, these two parts are clearly the same as the longitudinal and transverse parts because

\[- \nabla \phi = -ik \phi \text{ is longitudinal} \]

and if \( \nabla \cdot \vec{A} = 0 \) (then \( \nabla \cdot \vec{A} = 0 \) (W.l.o.g.))

i.e. \( k \cdot \vec{A} = 0 \) so \( \vec{A} \) is transverse.

'Electrostatic' waves are those that are describable by the electrostatic part of the electric field, which is the longitudinal part: \( |E_l| \gg |E_t| \)

If we simply say \( E_t = 0 \), then the dispersion relation becomes \( \varepsilon \cdot E_t = 0 \). This is not the correct dispersion relation for electrostatic waves. It is too restrictive. In general, there is a more significant way in which to get solutions where \( |E_l| \gg |E_t| \). It is for \( N^2 \) to be very large compared to all the components of \( \varepsilon : N^2 \gg |\varepsilon| \).

If this is the case, then the dispersion relation is approximately

\[ N^2 E_t = \varepsilon \cdot E_t \ ; \]

\( E_t \) is small but not zero.

We can then annihilate the \( E_t \) term by taking the \( N \) component of this equation, leaving

\[ N \cdot \varepsilon \cdot E_t = (N \cdot \varepsilon \cdot N) E_t = 0 \ ; \ k \cdot \varepsilon \cdot k = 0 \ . \]