The Fusion Problem.

Electrically charged nuclei repel each other by Coulomb force. Has to be overcome for fusion to occur.

\[ V_B \propto \frac{1}{r} \]

Potential \[ V_B \]

Coulomb

Strong-Force Binding.

\[ V_B \approx \frac{e^2}{4\pi \varepsilon_0 r_n} \]

\[ r_n \approx 2 \, (\text{Nuclear Radius}) \approx 2 \times 10^{-15} \, \text{m} \]

\[ \Rightarrow \quad V_B \approx 0.5 \, \text{MeV} \quad (\text{Hydrogen}) \]
Classically we would require a threshold energy of (e.g.) deuteron \( \approx 0.5 \text{ MeV} \) before fusion occurs.

Quantum Mechanics allows tunnelling through the potential barrier at (much) lower energy.

However the probability of tunnelling can be very small.
Calculate the Tunnelling Coefficient.

Schrödinger Equation:

\[
\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) \right] \psi = E \psi
\]

Hamiltonian \hspace{1cm} Energy

Spherical coordinates

\[
\nabla^2 \rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r \sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}
\]

Seek separable solutions

\[\rightarrow \text{angular parts}: \text{Spherical Harmonics } Y^m_l\]

Substitute \( \psi(r) = \frac{x(r)}{r} \), then

\[
\left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2} \right) + V(r) \right] x = E x
\]

\[\rightarrow \frac{p_r^2}{2m}, \hspace{0.5cm} \rightarrow \frac{p_\theta^2}{2m} = \frac{(\text{Angular Momentum})^2}{2m r^2}\]

Same equation as \text{Hydrogen Atom}.\]
Of course we are talking about motion of a deuteron in field of another. Mass is:

\[ m = \frac{m_D}{2} \quad (m_D \approx 2 \text{ mass of proton}) \]

Reduced mass: \( \frac{m_1 m_2}{m_1 + m_2} \); other D is not fixed.

Since the transverse momentum adds to the effective repulsion potential, consider only the case \( l = 0 \) (S wave) (Symmetric).

Equation:

\[ \left\{ -\frac{\hbar}{2m} \frac{d^2}{dr^2} + V(r) \right\} X = EX \]

or

\[ \frac{d^2}{dr^2} X = \frac{2m}{\hbar^2} (V(r) - E)X = K^2 X \]

(say)

For most of the relevant region \( V \) is Coulomb:

\[ V = \frac{e^2}{4\pi\epsilon_0 r} \]
Method of Solution (WKBJ)

Seek solution in form \( \chi(r) = a e^{-f(r)} \)

Substitute:
\[
-\frac{d^2f}{dr^2} + (\frac{df}{dr})^2 - K^2 = 0
\]

so, assuming \( f \) is slowly varying, ignore 2nd derivative:
\[
(\frac{df}{dr})^2 = K^2 \quad \text{i.e.} \quad f = \int K(r') \, dr'
\]

Tunnelling factor (by which \( \chi \) decreases in penetrating the barrier) is thus
\[
\exp \left\{ -\int_{r_n}^{r_c} \sqrt{\frac{2m}{\hbar^2} \left( \frac{e^2}{4\pi \varepsilon_0 r'} - E \right)} \, dr' \right\}
\]

where \( r_c \) is the point at which
\[
\frac{e^2}{4\pi \varepsilon_0 r_c} - E = 0.
\]

Need to do the integral.
• Make substitution \( x = \frac{r}{r_c} = r \left( \frac{4\pi \varepsilon_0 E}{e^2} \right) \)

then

\[
I = \frac{\sqrt{2m}}{\hbar} \frac{e^2}{4\pi \varepsilon_0 E_n} \int_x^1 (\frac{x}{x_n} - 1)^{\frac{1}{2}} \, dx
\]

• 2nd Substitution: \( x = \cos^2 u \); \( dx = -2 \sin u \cos u \, du \)

\[
(\frac{1}{x} - 1)^{\frac{1}{2}} \cos u = \sin u
\]

so

\[
I = \frac{\sqrt{2m}}{\hbar} \frac{e^2}{4\pi \varepsilon_0 E_n^2} \int_u^0 -2 \sin^2 u \, du
\]

In the case of interest \( T_n \ll r_c \Rightarrow x_n \ll 1 \)

\[
\Rightarrow u_n \approx \frac{\pi}{2}
\]

so \( \int_{u_n}^0 \approx \frac{\pi}{2} \)

and

\[
I \approx \frac{\sqrt{2m}}{\hbar} \frac{e^2}{4\pi \varepsilon_0} \frac{\pi}{2} \cdot \frac{1}{E_n^2}
\]

Tunnelling Factor, \( T = \exp(-I) = \exp\left(\frac{-B}{2E_n}\right) \)

Probability of tunnelling \( \propto T^2 = \exp\left(\frac{-B}{E_n}\right) \)
Cross-Section (Semi-Classical, Heuristic).

Impact Parameter

\[ o \rightarrow b \rightarrow o \]

Collision will correspond to angular momentum \( l = 0 \) only if

\[ bm\nu \leq \hbar \]

i.e. \( b \leq b_0 = \frac{\hbar}{m\nu} = \frac{1}{\left(\frac{1}{2}m\nu^2\right)^{\frac{1}{2}}} \frac{\hbar}{\sqrt{2m}} \]

\[ = \frac{\hbar}{\sqrt{2m}} \frac{1}{E^{1/2}} \]

Hence the collision cross-section for a fusion reaction is

\[ \sigma = \pi b_0^2 \cdot T^2 \]

Area of disc radius \( b_0 \)

Tunnelling Prob.

\[ = \frac{\pi \frac{\hbar^2}{2m}}{\frac{1}{E}} \exp \left( \frac{-B}{E^{1/2}} \right) \]

\[ [B = \pi \frac{\sqrt{2m}}{\hbar} \frac{e^2}{4\pi \epsilon_0}] \]

"Gamow Cross-Section"

(1928!!)
Evaluate \( B = \frac{\pi}{4} \frac{\sqrt{2} \times 1.67 \times 10^{-27}}{1.06 \times 10^{-34}} \frac{(1.6 \times 10^{-19})^2}{8.85 \times 10^{-12}} \)
\[ = 3.96 \times 10^{-7} \text{ J}^2 \]

Or (so we can measure \( E \) in keV)

\[ B = \frac{3.96 \times 10^{-7}}{(1.6 \times 10^{-16})^2} = 31.3 \text{ (keV)}^{1/2} \]

Notice that the exponent becomes very large as we go to low energy:

\[ E \quad 2I = \frac{B}{E^{1/2}} \quad T^2 = \exp\left(-\frac{B}{E^{1/2}}\right) \]

<table>
<thead>
<tr>
<th>( E )</th>
<th>( 2I )</th>
<th>( T^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 keV</td>
<td>9.9</td>
<td>( 5 \times 10^{-5} )</td>
</tr>
<tr>
<td>1 keV</td>
<td>31.3</td>
<td>( 2.6 \times 10^{-14} )</td>
</tr>
<tr>
<td>0.1 keV</td>
<td>99</td>
<td>( 1 \times 10^{-43} )</td>
</tr>
<tr>
<td>0.01 keV</td>
<td>313</td>
<td>( 9 \times 10^{-135} ) (!!)</td>
</tr>
</tbody>
</table>

This exponential dependence dominates.

\[ \frac{\pi k^2}{2m} \approx 600 \text{ keV-barn}; \text{ but } \approx 100 \text{ keV-barn is better estimate} \]
Simple-Minded Calculation of Reaction Rate.

Suppose we have density $n$ of deuterons ($m^{-3}$) moving at velocity $v$ through a background of stationary deuterons of density also $n$.

```
  n   n
  "projectiles" "targets"
```

Rate per unit volume at which reactions occur is

$$R = (n^2 \sigma v) \text{ m}^{-3} \text{ s}^{-1}$$

Note that $E = \frac{1}{2}mv^2$ so

$$R = n^2 \sigma(E) \left(\frac{2E}{m}\right)^{1/2}$$

$$= \frac{\pi \hbar^2}{2^{1/2}m^{3/2}} \frac{1}{E^{1/2}} \exp\left(-\frac{B}{E^{1/2}}\right) n^2$$

Example (to be generous):

take solid density $n \approx 10^{30} \text{ m}^{-3}$;
Evaluate $\frac{\pi h^2}{\sqrt{2} \ m^3/s} = 3.7 \times 10^{-28} \ J m^3/s = 3 \times 10^{-20} (\text{keV})^{1/2} m^3/s$

Realize that power density from fusion reactions is reaction rate ($R$) ($m^{-3} s^{-1}$) times energy yield $Q$ per reaction ($\sim 3.6 \text{ MeV} \approx 5.8 \times 10^{-13} J$) so we get:

<table>
<thead>
<tr>
<th>Energy (keV)</th>
<th>$R = \frac{\pi h^2}{\sqrt{2} \ m^3/s} \frac{n^2}{E_n} T^2$</th>
<th>$R Q$ (W/m$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 keV</td>
<td>$8 \times 10^{26}$</td>
<td>$5 \times 10^{14}$</td>
</tr>
<tr>
<td>0.1 keV</td>
<td>$9 \times 10^{-3}$</td>
<td>$6 \times 10^{-15}$</td>
</tr>
<tr>
<td>0.01 keV</td>
<td>$3 \times 10^{-93}$</td>
<td>$2 \times 10^{-105}$</td>
</tr>
<tr>
<td>[0.001 keV]</td>
<td>$\sim 10^{-390}$</td>
<td>$\sim 10^{-402}$</td>
</tr>
</tbody>
</table>

Age of universe $= 10^{18}$ sec.

Although there is, in principle, a finite reaction rate at say $E = 10 \text{ eV}$ it would give utterly negligible probability of even one reaction during age of universe!
Energy Scales:

Room Temperature (300K) \(~0.03\) eV.

Typical molecular Binding Energy \(~3\) eV.

Typical Atomic Binding Energy \(~10\) eV.

Highest ion temperature in a Tokamak \(30\) keV.

Even if deuterons somehow acquired energy typical of atomic binding, fusion rate would be \textit{zip}.

Caution: If deuterons have Maxwell-Boltzmann distribution need to average \(\langle\sigma v\rangle\) over distribution. Leads to a slightly different form & numerical values. Conclusion is same.